Who cares about codensity monads?

QTCat, Hamburg

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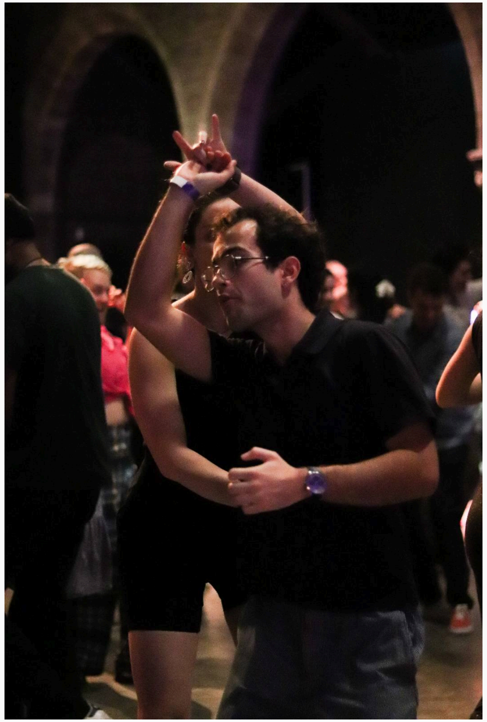

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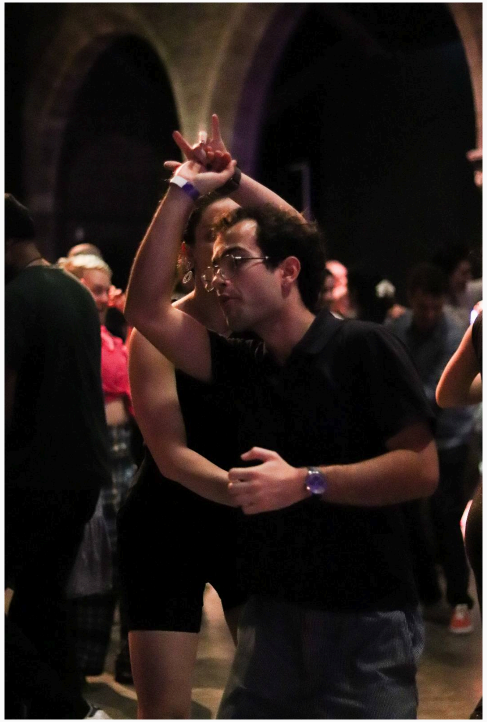

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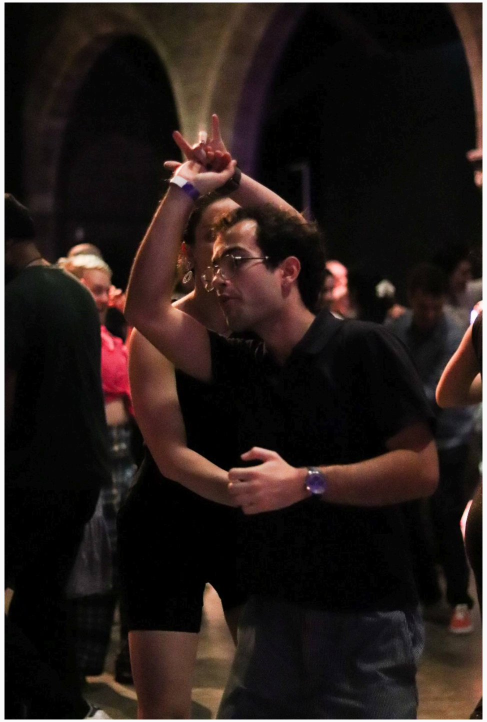


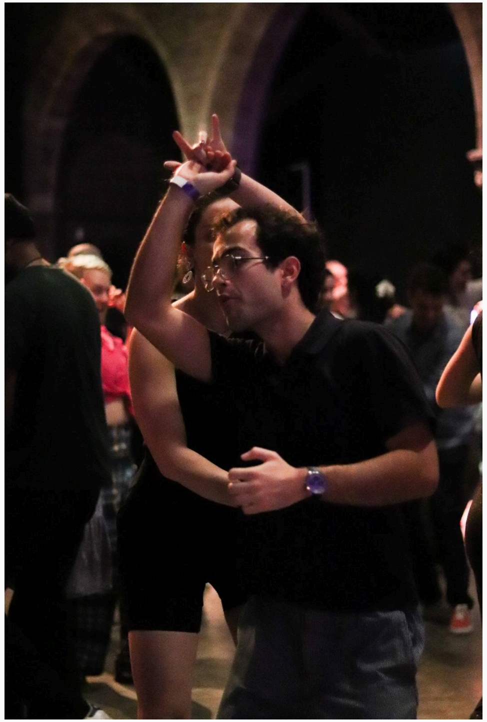
I do!

Hi! I’m **Adrián** (he/him).

I’m finishing my PhD in Edinburgh (my viva is next week).

I care about codensity monads.

I also care about many other things: music (especially piano), theatre, swing dancing, bouldering, video games, ...



Outline

1. The monadic reflection of a functor

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$$\begin{tikzcd}
\mathbf{Grp} \arrow[dr, bend right] & \mathbf{Ring} \arrow[d] & \mathbf{Vect}_k \arrow[dl, bend left] \\
& \mathbf{Set} &
\end{tikzcd}$$

1. The monadic reflection of a functor

$$
\begin{tikzcd}
\textcolor{orange}{\mathbf{FinSet}} \arrow[drrr, bend right, orange] & & \mathbf{Grp} \arrow[dr, bend right] & \mathbf{Ring} \arrow[d] & \mathbf{Vect}_k \arrow[dl, bend left] \\
& & & \mathbf{Set} &
\end{tikzcd}
$$

CAT / Set

$$
\begin{tikzcd}
\mathbf{FinSet} \arrow[drr, bend right] & \mathbf{Grp} \arrow[dr] & \mathbf{Ring} \arrow[l, dashed] \arrow[d] & \mathbf{Vect}_k \arrow[ll, dashed, bend right] \arrow[dl] \\
 & & \mathbf{Set} &
\end{tikzcd}
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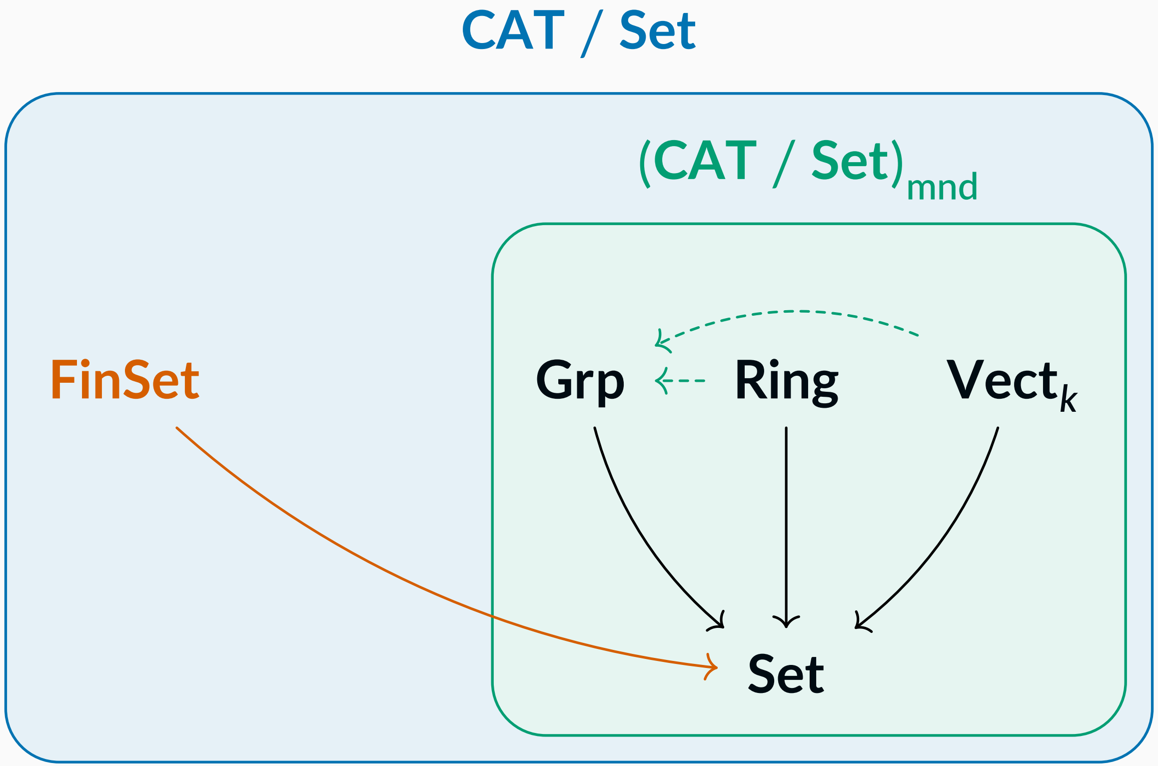
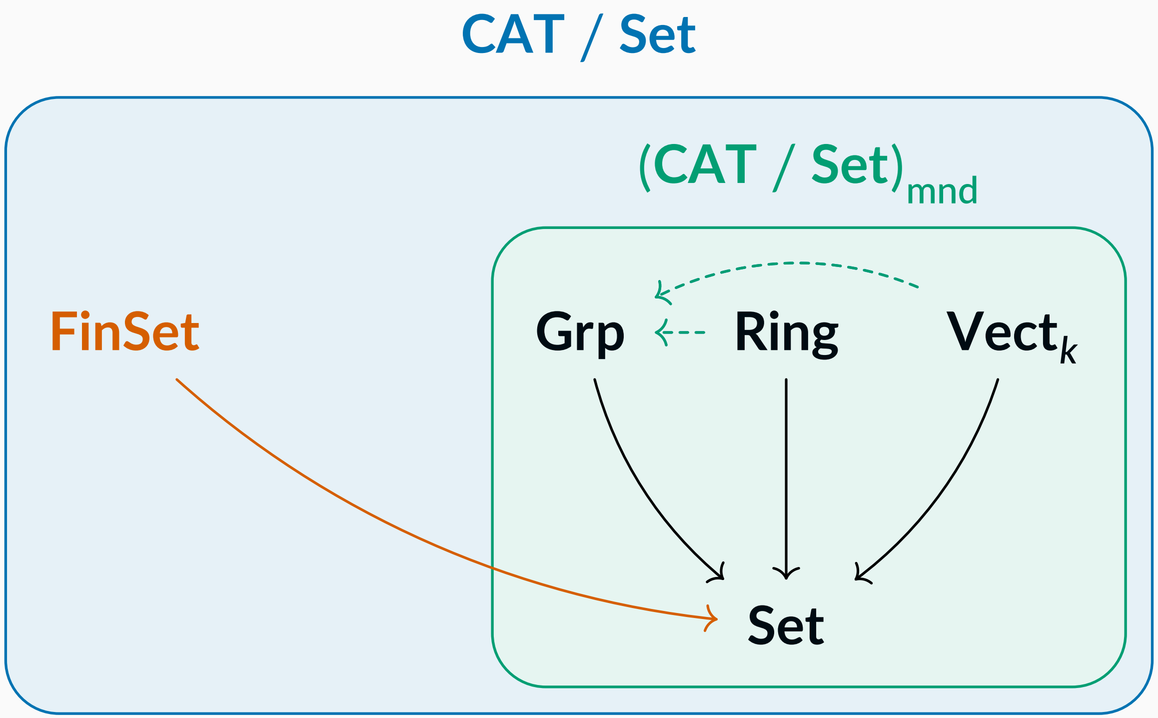
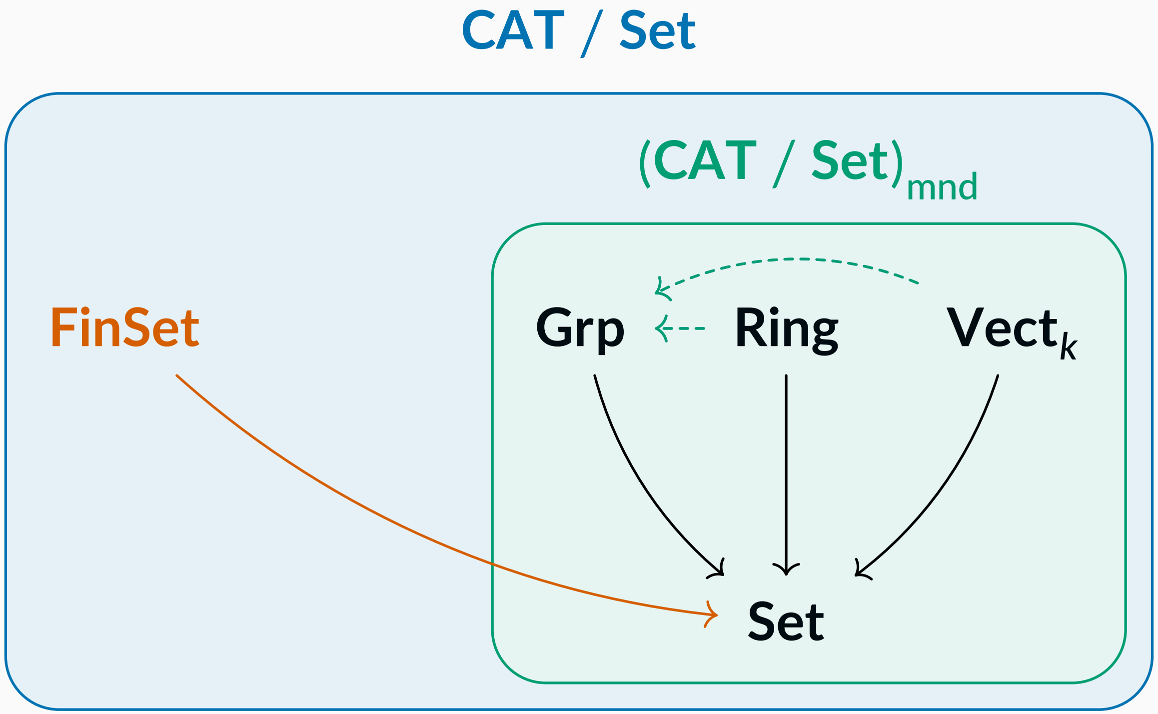

CAT / Set
(CAT / Set)$_{\mathrm{mnd}}$
FinSet
Grp
Ring
Vect$_k$
Set



What monadic functor (algebraic theory) best approximates **FinSet** $\to$ **Set**?

1. The monadic reflection of a functor



What monadic functor (algebraic theory) best approximates **FinSet** $\to$ **Set**?

In general: does the inclusion

$$
\begin{tikzcd}
(\mathbf{CAT} \,/\, \mathcal{C})_{\text{mnd}} \arrow[r, hook] & \mathbf{CAT} \,/\, \mathcal{C} \arrow[l, dashed, bend right]
\end{tikzcd}
$$

have a left adjoint?

$$\begin{array}{ccc} \mathbf{CAT}\,/\,\mathcal{C} & \dashrightarrow & (\mathbf{CAT}\,/\,\mathcal{C})_{\mathrm{mnd}} \\ \Cup & & \\ G\colon \mathcal{A} \to \mathcal{C} & & \end{array}$$

If $\mathrm{Ran}_G\, G$ exists[1], then it has a canonical monad structure: the **codensity monad** of G, denoted by T^G.

[1] E.g. if $\mathcal{A}$ is small and $\mathcal{C}$ is complete.

$$
\begin{tikzcd}
\mathbf{CAT} / \mathcal{C} \arrow[r, dashed] & (\mathbf{CAT} / \mathcal{C})_{\mathrm{mnd}} \\[-20pt]
\Cup & \Cup \\[-20pt]
G\colon \mathcal{A} \to \mathcal{C} \arrow[r, mapsto] & U^{T^G}\colon \mathcal{C}^{T^G} \to \mathcal{C}
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The forgetful functor $U^{T^G}\colon \mathcal{C}^{T^G} \to \mathcal{C}$ is the **monadic reflection** of G.

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$$\begin{tikzcd}
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If $G\colon\ \mathbf{FinSet} \hookrightarrow \mathbf{Set}$, then T^G is the **ultrafilter monad** $\mathbf{U}$, whose category of algebras is $\mathbf{CHaus}$. In other words, the algebraic theory* whose algebras are closest to finite sets is that of compact Hausdorff spaces!

$$
\begin{tikzcd}
\mathbf{FinSet} \arrow[r, "\mathrm{disc}"] \arrow[dr, hook, "G"'] & \mathbf{CHaus} \arrow[d, "U^{\mathbf{U}}"] \\
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$$

Some variants:

- Monads on **Set** "$\leftrightarrow$" *infinitary* single-sorted algebraic theories.
  We could also restrict to **finitary theories**[1].
  - The *finitary* monadic reflection of $\{2\} \to$ **Set** is **BA** $\to$ **Set**.
  - The *finitary* monadic reflection of **FinSet** $\hookrightarrow$ **Set** is just **Set** $=$ **Set**.

[1] E.g. groups, rings, but not suplattices or compact Hausdorff spaces.

[2] $\mathcal{E}$ with products and filtered colimits.

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  - The monadic reflection of **Field** $\to$ **CRing** is the free product completion of **Field**.

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- Other base categories.
  - The monadic reflection of **Field** $\to$ **CRing** is the free product completion of **Field**.
  - For suitable $\mathcal{E}$[2], the codensity monad of $\mathbf{FinFam}(\mathcal{E}^{\mathrm{op}}) \to \mathbf{Fam}(\mathcal{E}^{\mathrm{op}})$ is the ultraproduct monad. If $\mathbb{T}$ is a finitary algebraic theory and $\mathcal{E} = \mathbb{T}$-**Alg**, then the corresponding algebras are sheaves of $\mathbb{T}$-algebras on compact Hausdorff spaces.

[1] E.g. groups, rings, but not suplattices or compact Hausdorff spaces.
[2] $\mathcal{E}$ with products and filtered colimits.

2. Codensity presentations of monads

Every monadic functor is the monadic reflection of *some functor* (e.g. itself)
$\Leftrightarrow$ every monad T is the codensity monad of *some functor* (e.g. $U^T : \mathcal{C}^T \to \mathcal{C}$).

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What can we learn from knowing that a monad is a codensity monad?

Think of codensity monads as a way to give **presentations** of monads.

If we know that T is the codensity monad of a 'nice' functor G,
that's saying that T has a 'nice' presentation.

What counts as a ‘nice’ presentation or ‘nice’ functor?

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What counts as a ‘nice’ presentation or ‘nice’ functor?

- A **right adjoint** (boring).
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What counts as a 'nice' presentation or 'nice' functor?

- A **right adjoint** (boring).
  - If $F \dashv G$, the codensity monad of G is simply the monad GF induced by the adjunction.
- A **fully faithful functor**.
  - If $G\colon \mathcal{A} \hookrightarrow \mathcal{C}$ is fully faithful, then $\mathrm{Ran}_G\, G$ is a genuine extension of G.
  - Then $\mathcal{A}$ is a full subcategory of $\mathcal{C}^{T^G}$[1].

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  - Then $\mathcal{A}$ is a full subcategory of $\mathcal{C}^{T^G}$[1].
  - Today: a fully faithful presentation can tell us about existence and uniqueness of certain **distributive laws**!

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3. Distributive laws

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Let S and T be two monads on $\mathcal{C}$.

The composite endofunctor $TS\colon \mathcal{C} \to \mathcal{C}$ need not have a monad structure:

$$1_{\mathcal{C}} \xrightarrow{\eta^T \eta^S} TS \qquad\qquad TSTS \overset{?}{\dashrightarrow} TS$$

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Definition (Beck '67)

A **distributive law** of S over T is a natural transformation $\delta\colon ST \to TS$ compatible with the monad structures of S and T:

$$\begin{tikzcd} & T \arrow[dl, "\eta^S T"'] \arrow[dr, "T\eta^S"] & \\ ST \arrow[rr, "\delta"'] & & TS \end{tikzcd} \quad \begin{tikzcd} & S \arrow[dl, "S\eta^T"'] \arrow[dr, "\eta^T S"] & \\ ST \arrow[rr, "\delta"'] & & TS \end{tikzcd} \quad \begin{tikzcd} S^2T \arrow[r, "S\delta"] \arrow[d, "\mu^S T"'] & STS \arrow[r, "\delta S"] & TS^2 \arrow[d, "T\mu^S"] \\ ST \arrow[rr, "\delta"'] & & TS \end{tikzcd} \quad \begin{tikzcd} ST^2 \arrow[r, "\delta T"] \arrow[d, "S\mu^T"'] & TST \arrow[r, "T\delta"] & T^2S \arrow[d, "\mu^T S"] \\ ST \arrow[rr, "\delta"'] & & TS \end{tikzcd}$$

Theorem (Beck '67)

The following data are in bijection:

1. distributive laws $\delta \colon ST \to TS$,
2. monad lifts of T along $U^S \colon \mathcal{C}^S \to \mathcal{C}$, i.e. monads $\overline{T}$ on $\mathcal{C}^S$ with

$$TU^S = U^S\overline{T}, \qquad \eta^T U^S = U^S \eta^{\overline{T}}, \qquad \mu^T U^S = U^S \mu^{\overline{T}}.$$

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The eponymous example of a distributive law is of the free monoid monad **M** over the free abelian group monad **A** on **Set**:

$$\mathbf{M}\,\mathbf{A}\,\{x, y, z\} \ni x(y+z) \xmapsto{\ \delta\ } xy + xz \in \mathbf{A}\,\mathbf{M}\,\{x, y, z\}$$

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Using monadic reflection, we can turn an arbitrary functor into a monadic one.

What if we have a lift of T along an arbitrary functor $G\colon \mathcal{A} \to \mathcal{C}$?
Does this give a distributive law of T^G over T?

4. Distributive laws under codensity monads

Consider the situation:

- $S = \mathrm{Ran}_G\, G$ is the codensity monad of G
- $\overline{T}$ is a monad lift of T along G

lifts

$$\begin{tikzcd} \mathcal{A} \arrow[loop above, "\overline{T}"] \arrow[r, "G"] & \mathcal{C} \arrow[loop above, "T"] \arrow[loop below, "S"] \end{tikzcd}$$

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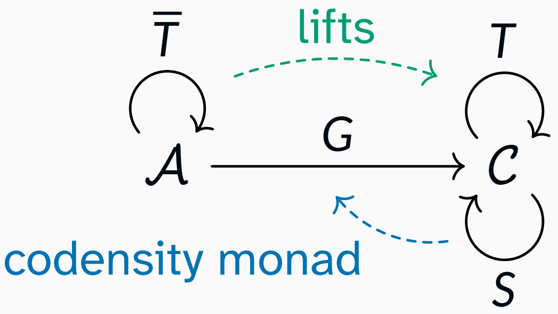

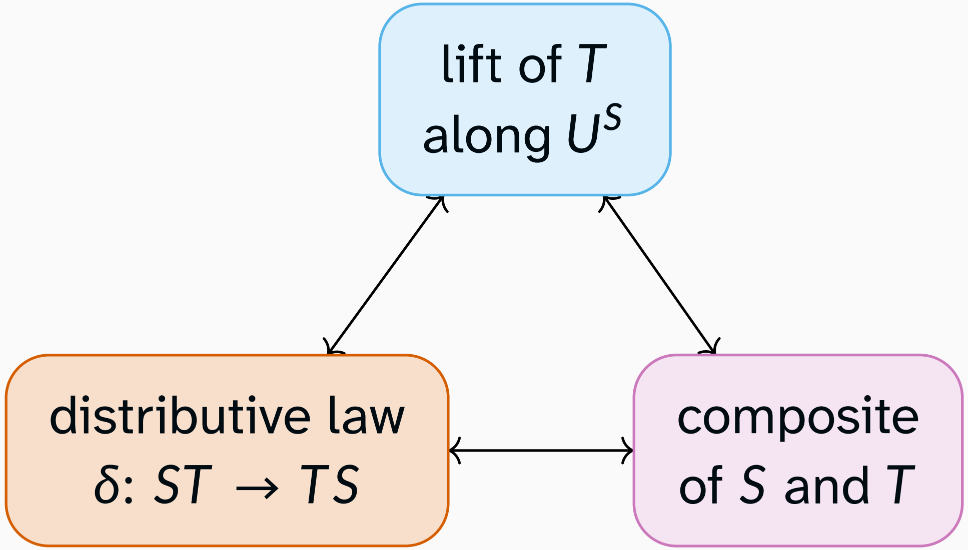

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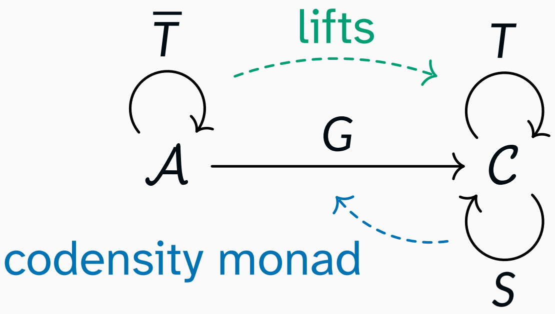

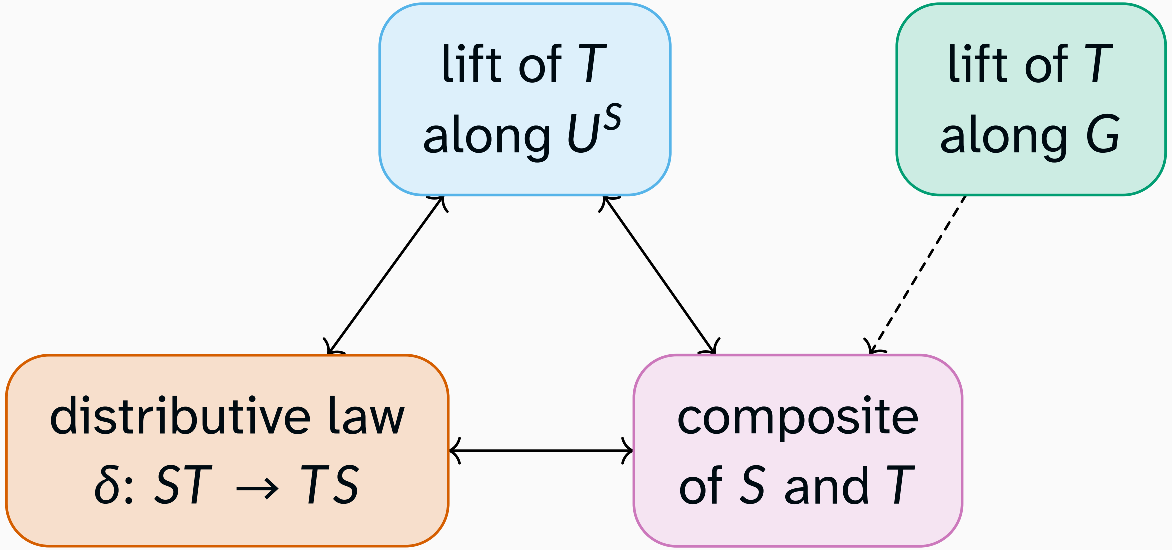

4. Distributive laws under codensity monads

lift of T along U^S → composite of S and T ← lift of T along G

Given a lift $\overline{T}$ along U^S, the composite monad is induced by the adjunction

$$\begin{tikzcd} \left(\mathcal{C}^S\right)^{\overline{T}} \arrow[r, bend left, "U^{\overline{T}}"] \arrow[r, phantom, "\top"] & \mathcal{C}^S \arrow[l, bend left, "F^{\overline{T}}"] \arrow[r, bend left, "U^S"] \arrow[r, phantom, "\top"] & \mathcal{C}. \arrow[l, bend left, "F^S"] \end{tikzcd}$$

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Given a lift $\overline{T}$ along G, we can take the **pushforward** of $\overline{T}$ along G, i.e.

$$G_\sharp \overline{T} = \mathrm{Ran}_G\, G\overline{T}.$$

It has a canonical monad structure. In fact, it is the codensity monad of

$$\mathcal{A}^{\overline{T}} \xrightarrow{U^{\overline{T}}} \mathcal{A} \xrightarrow{G} \mathcal{C}.$$

4. Distributive laws under codensity monads

Now, the pushforward $G_\sharp\overline{T} = \operatorname{Ran}_G G\overline{T}$ may well not be the composite of T and $S = \operatorname{Ran}_G G$.

$$\begin{tikzcd}
\mathcal{A} \arrow["\overline{T}", loop above] \arrow[rr, "G"] & & \mathcal{C} \arrow["T", loop above] \arrow["S"', loop below] \arrow["G_\sharp\overline{T}", loop right]
\end{tikzcd}$$

lifts

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However, it is a kind of *combination* of T and S, in the following sense:

[1]Monad maps $T \to S$ correspond to functors $\mathcal{C}^S \to \mathcal{C}^T$ over $\mathcal{C}$.

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Now, the pushforward $G_\sharp \overline{T} = \operatorname{Ran}_G G\overline{T}$ may well not be the composite of T and $S = \operatorname{Ran}_G G$.

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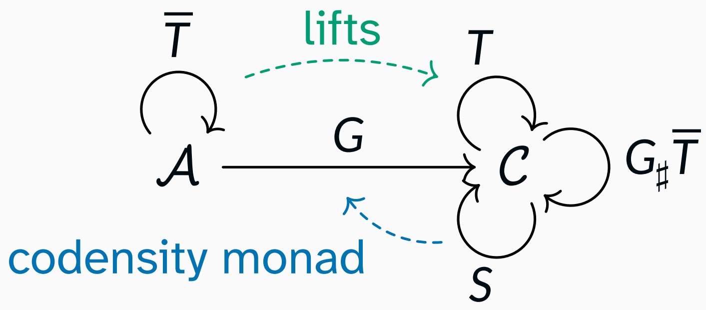

Lemma

There are canonical natural transformations

$$S \longrightarrow G_\sharp \overline{T}, \qquad T \longrightarrow G_\sharp \overline{T}, \qquad TS \longrightarrow G_\sharp \overline{T}.$$

- The first two are monad maps[1].
- If $\delta\colon ST \to TS$ is a distributive law **compatible** with G and $\overline{T}$, then so is the last.

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Compatibility means that the following diagram commutes:

$$\begin{tikzcd}
STG \arrow[rr, "\delta"] \arrow[d, equal] & & TSG \arrow[d, "\varepsilon"] \\
SG\overline{T} \arrow[r, "\varepsilon\overline{T}"] & G\overline{T} \arrow[r, equal] & TG
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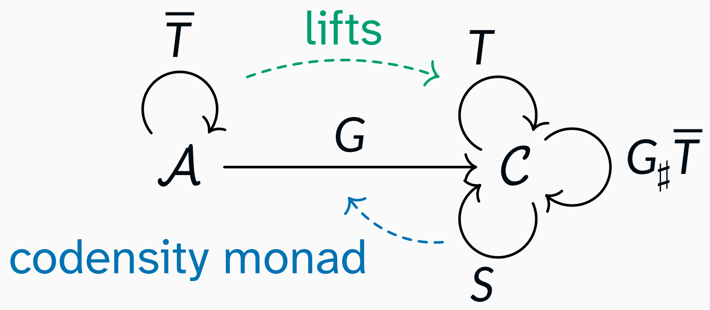
We shouldn't expect just any distributive law to be so compatible. After all, G is a *presentation* of S. However:

- If G is fully faithful, then any δ is compatible with G and $\overline{T}$.



Theorem

If the canonical map $\nu \colon TS \to G_\sharp\overline{T}$ is monic, then there is a (unique) distributive law $ST \to TS$ compatible with G and $\overline{T}$ iff the monad structure of $G_\sharp\overline{T}$ restricts along ν[1].

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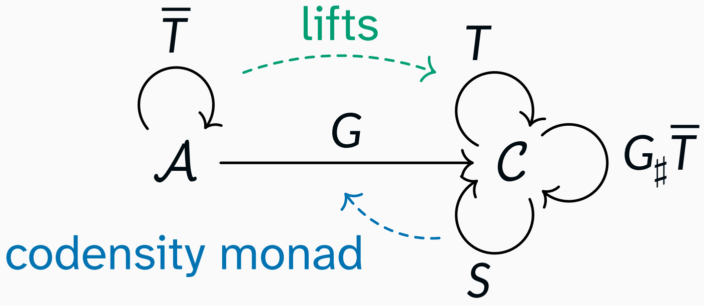
[1] Explicitly, if $\mu \cdot \nu\nu$ and η factor through ν.



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In particular, if **G is fully faithful** and **T preserves $\mathrm{Ran}_G\, G$**, then there is a **unique distributive law $ST \to TS$**.

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4. Distributive laws under codensity monads

Example

Let $G\colon$ **FinSet** $\hookrightarrow$ **Set** and T be a **polynomial monad** on **Set** which lifts along G, i.e. which sends finite sets to finite sets.

Then there is a unique distributive law of the ultrafilter monad over T.

Follows from the theorem, because polynomial endofunctors on **Set** preserve connected (and hence cofiltered) limits, so they preserve $\mathrm{Ran}_G\, G$.

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Let $G\colon$ **FinSet** $\hookrightarrow$ **Set** and T be a **polynomial monad** on **Set** which lifts along G, i.e. which sends finite sets to finite sets.

Then there is a unique distributive law of the ultrafilter monad over T.

Follows from the theorem, because polynomial endofunctors on **Set** preserve connected (and hence cofiltered) limits, so they preserve $\mathrm{Ran}_G\, G$.

E.g. we can take T to be

- the **exception monad** $(-) + E$ for any finite set E,
- the **monoid action monad** $M \times (-)$ for any finite monoid M,
- the **reader monad** $(-)^S$ or the **state monad** $(S \times (-))^S$ for any finite set S.

Example

The first part of the theorem can also be used to show that there no distributive laws of a given kind. E.g. with just a little work you can show that there is **no distributive law** of the ultrafilter monad over the **finite powerset monad**.

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Example (of something else)

One can play a similar game with **weak distributive laws** instead of normal distributive laws. These are similar to distributive laws, but one of the unit axioms is dropped.

4. Distributive laws under codensity monads

Example

The first part of the theorem can also be used to show that there no distributive laws of a given kind. E.g. with just a little work you can show that there is **no distributive law** of the ultrafilter monad over the **finite powerset monad**.

Example (of something else)

One can play a similar game with **weak distributive laws** instead of normal distributive laws. These are similar to distributive laws, but one of the unit axioms is dropped.

An analogous (but trickier) theorem can be used to recover the **weak** distributive law of the ultrafilter monad over the **powerset monad** constructed by Garner.

Summary

1. A functor $G\colon \mathcal{A} \to \mathcal{C}$ is a **presentation** for a monad on $\mathcal{C}$, i.e. its **codensity monad** $\mathrm{Ran}_G\, G$.
2. If G is **fully faithful**, then it is a specially nice presentation.
3. Such a presentation tells us about existence and uniqueness of certain **distributive laws** under $\mathrm{Ran}_G\, G$.

Some references

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