Cauchy density and pushforward monads

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Parts of Chapters 4 and 6 have been published in [24], although they have been partially rewritten and expanded on in this thesis.

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To George Seurat and Kilgore Trout

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Abstract

This thesis comprises two parts whose unifying thread is the concept of categorical density. The overarching theme is understanding what structure and properties of a category can be extended to a larger category inside of which the first one sits.

The first guise this idea takes is as a categorical generalisation of the concept of density for metric spaces. Following Lawvere's insight, which sees a metric space as a category enriched in the quantale of nonnegative reals, we study the enriched functors corresponding to maps whose image is topologically dense. These are the Cauchy dense functors. We explain how fully faithful Cauchy dense functors are intimately linked to Morita equivalence. We also give characterisations of Cauchy density in several contexts, and study the case of Cauchy dense functors between one-object enriched categories, i.e. monoids.

The second, and more substantial, undertaking of this thesis is investigating the construction of pushforward monads introduced in Street's formal theory of monads. Pushforward monads give a universal way of transporting a monad along a functor subject to the existence of a right Kan extension. They generalise codensity monads, which are the pushforwards of identity monads.

We study the properties of pushforward monads in a 2-category. In the 2-category of categories, pushforwards along a fully faithful functor are an important case, because the induced functor between categories of monads is itself fully faithful and has a partial left adjoint. In particular, every monad on a suitable full subcategory is the restriction of a monad on the entire category.

This highlights the question: if a monad T admits a lift along a functor G, what is the pushforward of this lift? We argue that the resulting monad deserves to be thought of as a generalisation of the composite monad induced by a distributive law of the codensity monad of G over T. We give conditions for when the pushforward is in fact the (weak) composite of such a (weak) distributive law.

Finally, we provide numerous examples of pushforward monads. These include pushforwards along the inclusions of the category of finite sets into that of all sets, of the category of fields into that of commutative rings, and of the category of finite families in a category into that of all families.

Lay Summary

A category is a mathematical device which consists of a collection of 'things' called objects and information on how these objects relate to each other. These relations cannot be entirely arbitrary: the way that A relates to B, together with the way that B relates to C, influence the way that A relates to C. The most fundamental example of a category is that whose objects are sets and where the information of how A relates to B is the collection of functions from A to B. Almost eerily, the idea of a category turns out to accurately describe the behaviour of many 'things' across mathematics. This gives a common ground to such different fields as algebra, topology and geometry.

Since categories emphasise the importance of relations between objects, it is natural to ask in what ways two categories can be related. This gives the notion of a functor from a category $\mathcal{A}$ to a category $\mathcal{B}$, which assigns an object of $\mathcal{B}$ to each object of $\mathcal{A}$ preserving those relations in $\mathcal{A}$.

A monad is a gadget one can attach to a category which allows for the consideration of objects equipped with some kind of structure. For example, a certain monad might bring into focus those objects of a category which admit a notion of addition, such as the set of integers $\mathbb{Z}$ in the category of all sets. In the language of category theory, we say that $\mathbb{Z}$ is an algebra for that particular monad.

Now suppose that we have two categories $\mathcal{A}$ and $\mathcal{B}$, a functor G between them, and a monad T attached to $\mathcal{A}$, i.e. a kind of structure that the objects of $\mathcal{A}$ can carry. What kind of structure can the objects of $\mathcal{B}$ carry which would make them most closely resemble the algebras for our monad T? The answer is given by the pushforward of the monad T along the functor G. This is the canonical monad generated by T which can be attached to $\mathcal{B}$ through G. It might be quite a complex monad, but the fact that it is generated by T provides a first step to understanding it. As an example, we could ask: what structure can sets carry which would make them most closely resemble finite sets? The answer turns out to be the structure of a particular kind of continuous space, called a compact Hausdorff space. This is remarkable, because finite sets are far from being representative examples of compact Hausdorff spaces such as spheres or tori.

In this thesis, we study the construction of pushforward monads in various levels of generality. We use it as a tool to prove properties about complex monads from simpler monads which generate them. We show how several important monads are in fact generated as pushforwards. Separately, we also consider a particular way in which two categories can be closely related, called a Cauchy dense functor, which is itself inspired by the topological notion of a dense subset.

Contents

Chapter 1

Introduction

Kan extensions are a ubiquitous tool in category theory. They provide universal answers to the question: how can a functor $\mathcal{A} \to \mathcal{C}$ be extended along a functor $\mathcal{A} \to \mathcal{B}$ to a functor $\mathcal{B} \to \mathcal{C}$? This simple question, when considered in the right contexts, is able to encapsulate surprising amounts of category theory, prompting Mac Lane to famously argue that 'all concepts are Kan extensions' [48, §X.7]. It is therefore natural that most of the contents of this thesis depend on this construction.

We will use Kan extensions to understand what structure on a category $\mathcal{B}$ is actually determined by, or can be induced by, structure on a category $\mathcal{A}$ through a functor $\mathcal{A} \to \mathcal{B}$. The actual structure in question will shift throughout the thesis. In Chapter 3 on Cauchy density, the structure we consider is that of a functor into a third category $\mathcal{C}$. In Chapters 4, 5 and 6 on pushforward monads, the structure will be that of a monad on $\mathcal{A}$.

1.1 Cauchy density

Among Hausdorff spaces, a continuous map $f\colon A \to B$ whose image is dense is an epimorphism: two continuous maps $g, h\colon B \to C$ which agree on a subset must also agree on its closure. One can transfer this topological idea into category theory using Lawvere's insight of viewing metric spaces as enriched categories [43].

He realised that a category X enriched in the monoidal poset of nonnegative extended reals $\mathbb{R}^+ = ([0, \infty], \geq, +, 0)$ ought to be thought of as generalised metric space. Indeed, to each pair of points $a, b \in X$, one assigns a distance $X(a, b) \in \mathbb{R}^+$ from a to b. Composition corresponds to the triangle inequality

$$X(a, b) + X(b, c) \geq X(a, c),$$

and identities give $X(a, a) = 0$. This notion differs from that of a classical metric space in the sense that distances may be infinite (because $\infty \in [0, \infty]$), different points might be distance 0 apart, and, most importantly, distances may not be symmetric ($X(a, b) \neq X(b, a)$ in general). An $\mathbb{R}^+$-enriched category is commonly known as a **Lawvere metric space**. An $\mathbb{R}^+$-functor is a **short map** of metric spaces, also called a distance-decreasing or 1-Lipschitz map, i.e. a function $f\colon X \to Y$ such that $X(a, b) \geq X(fa, fb)$.

Note that $\mathbb{R}^+$ is good base for enrichment because it is a complete and cocomplete, symmetric monoidal closed category. Such a category is commonly called a **Bénabou cosmos**. Lawvere specialised many concepts of enriched category theory to the $\mathbb{R}^+$ case to recover important notions of metric geometry. For instance, the Yoneda embedding becomes the isometric embedding of a metric space X into the space of $\mathbb{R}^+$-valued short maps with the sup metric, and the space of closed subsets of X equipped with a nonsymmetric version of the Hausdorff metric is naturally a subspace of the space of presheaves on X.

Through this programme, Lawvere highlighted that many of the basic notions of metric geometry did not depend on the axioms of a classical metric space beyond those of a Lawvere metric space, i.e. finiteness, separation and symmetry. Thus, enriched category theory can help inform metric geometry. Perhaps more importantly, though, he showed that this relationship extends both ways: metric geometry can help inform enriched category theory.

The foremost example of this is his categorical characterisation of the completion of a metric space X, i.e. the set of equivalence classes of Cauchy sequences in X. He showed that this is exactly the completion of X under those $\mathbb{R}^+$-colimits weighted by an absolute weight. This notion makes sense regardless of the base of enrichment, and gives the definition of the **Cauchy completion** $\overline{\mathcal{C}}$ of a $\mathcal{V}$-category $\mathcal{C}$. This process turns out to be extremely meaningful: if $\mathcal{V}$ is a Bénabou cosmos and $\mathcal{A}$ and $\mathcal{B}$ are small $\mathcal{V}$-categories, then $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \simeq [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ iff $\overline{\mathcal{A}} \simeq \overline{\mathcal{B}}$. A proof can be found e.g. in Kelly's book [33, Prop. 5.28].

As well as the completion of a metric space, Lawvere generalised the notion of a short map with dense image (in the topological sense) to $\mathcal{V}$-functors. A $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ is **Cauchy dense** if the map

$$\int^{a\in\mathcal{A}} \mathcal{B}(Fa, b') \otimes \mathcal{B}(b, Fa) \longrightarrow \mathcal{B}(b, b') \tag{1.1}$$

induced by composition in $\mathcal{B}$ is an isomorphism for all $b, b' \in \mathcal{B}$. The reader familiar with $\mathcal{V}$-profunctors will recognise this as the condition that the counit of the profunctor adjunction $F_* \dashv F^*$ be invertible. Lawvere called such functors $\mathcal{V}$-dense, but the word dense has since taken a different (although related) meaning in category theory. Because of this, we use the term 'Cauchy dense' instead, following Day [17, p. 428].

Because of the added categorical structure, Cauchy dense functors are not quite epimorphisms of $\mathcal{V}$-categories, as is the case for classical metric spaces. Instead, they are the **cofully faithful** 1-cells of the 2-category $\mathcal{V}$-Cat, also called lax epimorphisms, i.e. those $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{B}$ such that precomposing with F is a fully faithful (ordinary) functor $\mathcal{V}\text{-}\mathsf{Cat}(\mathcal{B}, \mathcal{C}) \to \mathcal{V}\text{-}\mathsf{Cat}(\mathcal{A}, \mathcal{C})$ for all $\mathcal{C}$. In fact, Lucatelli Nunes and Sousa [47, Thm. 5.6 and 5.11] showed:

Theorem. Let $\mathcal{V}$ be a Bénabou cosmos. The following are equivalent for a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ between small $\mathcal{V}$-categories:

(i) F is Cauchy dense;

(ii) F is cofully faithful in $\mathcal{V}$-Cat;

(iii) the $\mathcal{V}$-functor $[F, \mathcal{C}] \colon [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$ is fully faithful for all $\mathcal{C}$;

(iv) F is absolutely dense;

(v) F is absolutely codense.

For a functor F to be **absolutely dense** means that $(1_{\mathcal{B}}, 1_F)$ is a pointwise left Kan extension of F along itself and, moreover, that this extension is preserved by any $\mathcal{V}$-functor. If $(1_{\mathcal{B}}, 1_F)$ is an absolute right Kan extension instead, then F is **absolutely codense**. It follows, in particular, that every Cauchy dense functor is dense and codense.

In Chapter 3, we revisit this result and study the properties of Cauchy dense $\mathcal{V}$-functors. We provide a streamlined proof of the theorem above using the proarrow equipment $(-)_* \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$ (Theorems 3.1.6 and 3.1.8). Restricting to the fully faithful Cauchy dense functors, we obtain a universal property of the Cauchy completion of a small $\mathcal{V}$-category $\mathcal{A}$: it is the largest $\mathcal{V}$-category that admits a fully faithful Cauchy dense functor from $\mathcal{A}$. (Theorem 3.2.5). For $\mathcal{V} = \mathbb{R}^+$, this recovers the fact that the completion of a space X is the largest space into which X embeds isometrically as a dense subset. This is analogous to a result by Avery and Leinster [8, Cor. 8.3 and Thm. 8.4] which gives a similar characterisation of the reflexive completion $\mathcal{R}(\mathcal{A})$ of a small $\mathcal{V}$-category $\mathcal{A}$ as the largest category into which $\mathcal{A}$ admits a fully faithful, dense and codense functor.

In turn, this gives a characterisation of fully faithful Cauchy dense functors akin to that in part (iii) of the theorem above: a $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ is fully faithful and Cauchy dense iff $[F, \mathcal{C}] \colon [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$ is an equivalence for every Cauchy complete $\mathcal{C}$ (Theorem 3.2.7). This result is essentially already present in work of Lucatelli Nunes, Prezado and Sousa [53, Thm. 2.5], but was obtained independently by the author.

The remainder of Chapter 3 focuses on studying the behaviour of Cauchy dense $\mathcal{V}$-functors for a number of $\mathcal{V}$. An important special case is given by Cauchy dense maps of monoids, seen as one-object $\mathcal{V}$-categories. For $\mathcal{V} = \mathsf{Set}$ and $\mathcal{V} = \mathsf{Ab}$, we show that they are exactly the epimorphisms of monoids. This does not hold in general however, and we provide a counterexample where $\mathcal{V}$ is the category of finitary endofunctors of Set.

Lastly, we study whether the Cauchy density condition can be simplified to only requiring that the map (1.1) be invertible when $b = b'$. We show that this is possible when $\mathcal{V}$ is a preorder, as is the case for $\mathcal{V} = \mathbb{R}^+$, or when $F \colon \mathcal{A} \to \mathcal{B}$ is split-full (Definition 3.2.2).

1.2 Pushforward monads

The second and more substantial part of this thesis focuses on the construction of pushforward monads. These were introduced by Street in his formal theory of monads [58], but have received little attention since. In Chapters 4, 5 and 6 we undertake an extensive study of the properties of this construction, its relation to other important notions in the theory of monads (such as distributive laws), and several examples of it. Chapter 4 and some parts of Chapter 6 are based on the article [24] published by the author.

Monads are a central notion in category theory which have enjoyed many applications in fields such as homological algebra, universal algebra and programming language semantics. For our purposes, we will mostly view them as an instrument for categorical universal algebra. We now give an overview of their role as such, building up to the motivation for the study of pushforward monads.

From algebraic theories to monads Classical universal algebra is concerned with structures defined by a set of abstract finitary operations subject to a set of universally quantified equations involving these operations. The classic example is the theory of groups, which can be presented by a binary operation m, a unary operation i and a nullary operation (i.e. a constant) e, subject to the universally quantified equations

$$\begin{gathered} m(x_1, m(x_2, x_3)) = m(m(x_1, x_2), x_3), \\ m(e, x_1) = x_1 = m(x_1, e), \\ m(i(x_1), x_1) = e = m(x_1, i(x_1)). \end{gathered}$$

Such a theory gives a notion of model: a set equipped with operations interpreting the abstract operations in the theory and satisfying all the equations. A model for the theory above is, of course, a group. Importantly, a theory also gives a notion of model homomorphism, which in our example is just a group homomorphism. The category of models and model homomorphisms $\mathbb{T}\text{-}\mathsf{Mod}$ for a theory $\mathbb{T}$ has a forgetful functor $U^{\mathbb{T}} \colon \mathbb{T}\text{-}\mathsf{Mod} \to \mathsf{Set}$, which is always monadic.

Thus, every theory $\mathbb{T}$ gives a monad on Set whose category of algebras is $\mathbb{T}\text{-}\mathsf{Mod}$. If all the abstract operations of $\mathbb{T}$ have finite arity, as in the example of groups above, then the corresponding monad is **finitary**, i.e. it preserves filtered colimits. In fact, there is a one-to-one correspondence between finitary theories and finitary monads on Set. Thanks to this, one obtains finitary monads on Set whose algebras are traditional objects of study in algebra: monoids, groups, rings, modules, Lie algebras, etc.

An analogous correspondence holds if the arity of the abstract operations in $\mathbb{T}$ is always less than some fixed regular cardinal κ: such theories correspond to monads on Set which preserve κ-filtered colimits. In general, however, not every monad on Set is induced by a theory with operations bounded by some cardinal. A good example of this is the theory of suplattices (or complete join-semilattices): to present it one needs an abstract operation of each possible arity α, which correspond to taking the supremum of a subset of cardinality α. The framework of monads on Set allows one to speak of theories which do not have a set of operations, but rather a proper class, so long as the operations of any given arity do form a set. Such monads are said to have **no rank** and are significantly less well-behaved than those arising from theories with bounded arities, but allowing them vastly increases the number of categories which are considered **algebraic**, i.e. which admit a monadic functor to Set.

On the other hand, it should be noted that some theories are too large to correspond to monads on Set. These are the theories with a proper class of operations for some fixed arity. The classical example of this is the theory of complete lattices, which has a proper class of ternary operations (see Johnstone's book [31, §I.4.7]).

The operations of a monad One way to understand monads on Set as theories is by considering their operations. If T is such a monad and n is a finite set, then Tn is the set of n-ary operations of the corresponding theory. Indeed, for any T-algebra (A, α), each $\theta \in Tn$ gives a map $A^n \to A$: the composite

$$A^n = \mathsf{Set}(n, A) \xrightarrow{T} \mathsf{Set}(Tn, TA) \xrightarrow{\mathrm{ev}_\theta} TA \xrightarrow{\alpha} A.$$

Each such θ does not just give a concrete operation for each T-algebra, but these combine into a global operation on the whole category of algebras, because every T-algebra map must commute with the operation induced by θ. Explicitly, each $\theta \in Tn$ gives a natural transformation $(U^T)^n \to U^T$ where $U^T \colon \mathsf{Set}^T \to \mathsf{Set}$ is the forgetful functor and $(U^T)^n$ is its (objectwise) n-th power. In fact, we have

$$Tn \cong [\mathsf{Set}^T, \mathsf{Set}](\mathsf{Set}^T(F^T n, -), U^T) \cong [\mathsf{Set}^T, \mathsf{Set}]\left((U^T)^n, U^T\right)$$

by the Yoneda lemma, because $(U^T)^n$ is the functor represented by $F^T n$, where $F^T \colon \mathsf{Set} \to \mathsf{Set}^T$ is the free T-algebra functor. This makes sense even if n is not finite, so that we can speak of operations of arbitrary arity. The data of the monad T hence contains that of the n-ary operations for each not-necessarily-finite set n. Once the set of n-ary operations is given for each n, what remains to specify the monad T, i.e. the functor structure, the unit and the multiplication, is exactly what it takes to specify how operations of different arities compose together to form other operations.

The passage from finitary monads to all monads on Set is partially reversible. Formally, the fully faithful inclusion $\mathsf{Mnd}^{\mathrm{f}}_{\mathsf{Set}} \hookrightarrow \mathsf{Mnd}_{\mathsf{Set}}$ of the category of finitary monads on Set to that of all monads on Set has a right adjoint $(-)^{\mathrm{f}}$. This is because the inclusion $J \colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$ is the free cocompletion of FinSet under filtered colimits, so that a functor $F \colon \mathsf{Set} \to \mathcal{A}$ preserves filtered colimits iff it is the left Kan extension $\mathrm{Lan}_J FJ$ of FJ along J. The coreflection $(-)^{\mathrm{f}}$ then sends a monad T on Set to the filtered-colimit-preserving endofunctor $\mathrm{Lan}_J TJ$ equipped with an induced monad structure. From the operations point of view, this process simply remembers the finitary operations of T, and generates the higher-arity operations in terms of those. We call T^{f} the **finitary part** of T.

The theory induced by a functor Given a category over Set, i.e. a functor $G \colon \mathcal{A} \to \mathsf{Set}$, one can ask to what extent $\mathcal{A}$ is algebraic. If G has a left adjoint $F \colon \mathsf{Set} \to \mathcal{A}$, then it induces a monad GF on Set and a comparison functor $K \colon \mathcal{A} \to \mathsf{Set}^{GF}$ over Set. This functor measures the failure of G to be monadic: K is an equivalence iff G is monadic. In fact, U^{GF} is the **monadic reflection** of G, in the sense that for any monad T on Set and functor H making the left diagram commute, there exists a unique functor $\widehat{H}$ making the right one commute:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "H"] \arrow[dr, "G"'] & & \mathsf{Set}^T \arrow[dl, "U^T"] \\ & \mathsf{Set}, & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Set}^{GF} \arrow[rr, dashed, "\widehat{H}"] \arrow[dr, "U^{GF}"'] & & \mathsf{Set}^T \arrow[dl, "U^T"] \\ & \mathsf{Set}. & \end{tikzcd} \tag{1.2}$$

If G does not have a left adjoint then it cannot be monadic. We can however still look for the theory it induces by finding its monadic reflection. Taking the operations point of view, the n-ary operations that G admits form the collection $T^G n$ of natural transformations $[\mathcal{A}, \mathsf{Set}](G^n, G)$. Note that in general this may be a proper class, in which case there is no hope to have T extend to an endofunctor of Set, let alone a monad. If nevertheless each $T^G n$ is a set, e.g. if $\mathcal{A}$ is small, then they assemble into a monad T^G on Set for which U^{T^G} is the monadic reflection of G in the sense above. This is the **codensity monad** of G. One can also consider the finitary part $(T^G)^{\mathrm{f}}$ of T^G, which we will call the **finitary codensity monad** of G. In this case, $U^{(T^G)^{\mathrm{f}}}$ is the finitary monadic reflection of G, i.e. it satisfies the analogous universal property in (1.2) with respect to finitary monadic functors.

If G is a right adjoint, then its codensity monad is GF. Thus, the forgetful functor $U\colon \mathsf{Grp} \to \mathsf{Set}$ induces, unsurprisingly, the theory of groups, as does its restriction to the full subcategory on the free groups. In fact, if we are only interested in the finitary operations, we can further restrict to the full subcategory of finitely generated free groups $\mathsf{Grp}_{\mathrm{ffg}}$. Let $D\colon \mathsf{Grp}_{\mathrm{ffg}} \hookrightarrow \mathsf{Grp}$ be the corresponding inclusion and $V = UD$. The finitary theory induced by V is again the theory of groups. Indeed, for a finite set n,

$$T^V n = [\mathsf{Grp}_{\mathrm{ffg}}, \mathsf{Set}](V^n, V) \cong [\mathsf{Grp}_{\mathrm{ffg}}, \mathsf{Set}](\mathsf{Grp}_{\mathrm{ffg}}(Fn, -), V) \cong VFn$$

by the Yoneda lemma, where $F\colon \mathsf{FinSet} \to \mathsf{Grp}_{\mathrm{ffg}}$ is the free group functor. The reader familiar with Lawvere theories will recognise $\mathsf{Grp}_{\mathrm{ffg}}$ as the opposite of the Lawvere theory of groups. In fact, the same argument shows that for any finitary monad T on Set, the finitary codensity monad of the forgetful functor $\mathsf{Set}^T_{\mathrm{ffg}} \to \mathsf{Set}$ is again T.

For another example, let $2\colon 1 \to \mathsf{Set}$ be the functor from the terminal category whose image is the two-element set 2. The finitary theory induced by this functor is commonly called the complete theory on the set 2. Its n-ary operations are all the functions $2^n \to 2$, which give the theory of Boolean algebras (see e.g. Lawvere's thesis [42, §III.1 Ex. 4]). The codensity monad of 2 is the double powerset monad, whose category of algebras is that of complete atomic Boolean algebras, equivalent to $\mathsf{Set}^{\mathrm{op}}$.

Lastly, consider the inclusion $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$. The finitary theory it induces is trivial, since $\mathsf{FinSet} = \mathsf{Set}_{\mathrm{ffg}}$. If we also consider the operations of J of infinite arity, however, we get a very different theory. Indeed, the codensity monad of J is the ultrafilter monad, as shown by Kennison and Gildenhuys [37, p. 341]. Its category of algebras was identified as the category CHaus of compact Hausdorff spaces and continuous maps by Manes [50, Prop. 5.5]. This gives a surprisingly simple universal algebraic explanation for the importance of compact Hausdorff spaces: they are the algebras for the theory induced by finite sets.

Monads on other categories Considering monads on other categories other than Set allows one to speak of algebraic structure carried not by sets, but by objects of an arbitrary category $\mathcal{C}$. For instance:

- Categories are graphs equipped with algebraic structure, since the functor $\mathsf{Cat} \to \mathsf{Graph}$ sending a category to its underlying graph is monadic. This well-known fact is proved e.g. by Barr and Wells in [9, p. 105]

- Banach spaces are complete metric spaces equipped with algebraic structure, since the functor $\mathsf{Ban}_1 \to \mathsf{CMet}_1$ sending a Banach space to its closed unit ball is monadic by Rosický's [54, Thm. 4.1].

- Finitary monads on Set are finitary endofunctors equipped with algebraic structure. This is true more generally for finitary monads on a locally finitely presentable category, as proved by Lack [38].

The interpretation of monads as theories determined by their operations is not immediately available in this setting (unless one asks for a strong monad on a closed category). However, one can still define the monad induced by a functor $G\colon \mathcal{A} \to \mathcal{B}$, i.e. the codensity monad T^G of G. Instead of considering sets of natural transformations between powers of G, we define $T^G = \mathrm{Ran}_G\, G$. If $\mathcal{B} = \mathsf{Set}$, this agrees with the previous definition whenever the Kan extension is pointwise. When T^G exists, we still have that U^{T^G} is the monadic reflection of G in the sense of (1.2) when Set is replaced by $\mathcal{B}$. If $\mathcal{B}$ is locally finitely presentable, we can also speak of the finitary codensity monad of G. As in the $\mathcal{B} = \mathsf{Set}$ case, this is the image of T^G under the right adjoint to the inclusion $\mathsf{Mnd}^{\mathrm{f}}_{\mathcal{B}} \hookrightarrow \mathsf{Mnd}_{\mathcal{B}}$.

Pushforward monads Suppose now that, in addition to a functor $G\colon \mathcal{A} \to \mathcal{B}$, we have a monad T on $\mathcal{A}$. Pushforward monads give a universal answer to the problem of transporting the monad T along the functor G. If $\mathrm{Ran}_G\, GT$ exists, then it has a canonical monad structure induced by T, which we call the **pushforward of T along G** and denote by $G_\sharp T$. Chapter 4 is devoted to the properties of this construction, both in the generality of monads in a 2-category and in the important case of monads in CAT.

In terms of monadic functors, $U^{G_\sharp T}$ is the monadic reflection of GU^T, so that $G_\sharp T$ is the codensity monad of the composite

$$\mathcal{A}^T \xrightarrow{U^T} \mathcal{A} \xrightarrow{G} \mathcal{B},$$

by Corollary 4.1.9. Therefore, pushforward monads are subsumed by the construction of codensity monads, at least in a 2-category which admits the construction of algebras in the sense of Street [58, p. 151]. On the other hand, the codensity monad of G is recovered as the pushforward of the identity monad.

One of the main reasons to study pushforward monads is that, beyond being examples of codensity monads, they carry a functorial structure. When all pushforward monads along $G\colon \mathcal{A} \to \mathcal{B}$ exists, they assemble into a functor

$$G_\sharp\colon \mathsf{Mnd}_\mathcal{A} \longrightarrow \mathsf{Mnd}_\mathcal{B}$$

by Lemma 4.1.4. If G is fully faithful, then so is $G_\sharp$, and in fact it admits a partial left adjoint defined for those monads on $\mathcal{B}$ which restrict along G (Theorem 4.3.11). In particular, the category of monads on a small full subcategory $\mathcal{A}$ of a complete category $\mathcal{B}$ is a full subcategory of the category of monads on $\mathcal{B}$. If $G\colon \mathcal{B}_{\mathrm{fp}} \hookrightarrow \mathcal{B}$ is the inclusion of the full subcategory of finitely presentable objects of a locally finitely presentable category, then there is another fully faithful functor $\mathsf{Mnd}_{\mathcal{B}_{\mathrm{fp}}} \hookrightarrow \mathsf{Mnd}_\mathcal{B}$ which sends T to $\mathrm{Lan}_G GT$. This less general inclusion is compared to the one given by pushforwards in Remark 4.3.16.

The existence of the partial left adjoint to $G_\sharp$ will follow from our work in Section 4.2, which gives an alternative universal property that pushforwards along $G\colon \mathcal{A} \to \mathcal{B}$ satisfy with respect to **G-determined monads** on $\mathcal{B}$, i.e. those monads T on $\mathcal{B}$ which are isomorphic to $\mathrm{Ran}_G TG$. This generalises a result of Adámek and Sousa [3, §6], by showing that if G is fully faithful, the codensity monad of G is the smallest G-determined submonad of a number of G-determined monads on $\mathcal{B}$ (Example 4.3.15(ii)).

In Section 4.3.3, we study the connection between codensity monads and free completions under classes of limits. This allows us to relate codensity monads to Diers's multiadjunctions [23, p. 58] and Tholen's $\mathfrak{D}$-pro-adjunctions [61, p. 148]. In particular, we use Diers's theory to identify the category of algebras of the codensity monad of the inclusion $\mathsf{Field} \hookrightarrow \mathsf{CRing}$ as the free product completion of Field. An alternative explicit proof of this fact was independently discovered by the author, and is presented later in Section 6.3.

Pushforward monads and weak distributive laws The composite of two endofunctors is again an endofunctor. However, the composite of two monads does not carry a canonical monad structure. Distributive laws were introduced by Beck [10] as a way to specify a monad structure on the composite endofunctor. Let T and S be monads on a category $\mathcal{A}$. A **distributive law** of T over S is a natural transformation $\delta\colon TS \to ST$ satisfying certain compatibility conditions with the units and multiplications of T and S (Definition 2.3.11).

Given such a δ, the endofunctor ST can be equipped with a monad structure with unit and multiplication given by

$$1_\mathcal{A} \xrightarrow{\eta^S\eta^T} ST \qquad \text{and} \qquad STST \xrightarrow{S\delta T} S^2T^2 \xrightarrow{\mu^S\mu^T} ST.$$

Moreover, one can define a monad $\overline{S}$ on $\mathcal{A}^T$ which lifts S along U^T by sending the T-algebra (A, α) to SA equipped with the T-algebra structure map

$$TSA \xrightarrow{\delta_A} STA \xrightarrow{S\alpha} SA.$$

In fact, Beck showed that distributive laws are in bijection with

(i) monad structures on ST such that $\eta^S T\colon T \to ST$ and $S\eta^T\colon S \to ST$ are monad maps, and satisfying the middle unitary law (Theorem 2.3.13);

(ii) monad lifts $\overline{S}$ of S along the monadic functor $U^T \colon \mathcal{A}^T \to \mathcal{A}$ (Theorem 2.3.14).

In the context of pushforward monads, it is natural to relax (ii) to a lift $\overline{S}$ of a monad S along an arbitrary functor $G \colon \mathcal{A} \to \mathcal{B}$. The pushforward $G_\sharp \overline{S}$ always comes equipped with two monad maps $G_\sharp 1 \to G_\sharp \overline{S}$ and $S \to G_\sharp \overline{S}$, where $G_\sharp 1$ is the codensity monad of G, satisfying an analogue of the middle unitary law. This situation is formally similar to (i), and leads to the natural question: when is $G_\sharp \overline{S}$ the composite monad induced by a distributive law of $G_\sharp 1$ over S?

This question is investigated in Chapter 5. The answer is affirmative when S preserves the right Kan extension of G along itself. In fact, in many cases this suffices to prove that there exists a unique distributive law of a given codensity monad over another monad (Theorem 5.1.5). Conversely, this characterisation can also be used to give a simple condition for the nonexistence of a distributive law of the same type (Proposition 5.1.8).

In general, asking that $G_\sharp \overline{S}$ be the composite induced by a distributive law is quite a strong requirement. Because of this, we also study the case of weak distributive laws, as introduced by Böhm [14] and Garner [28]. In fact, we find that the notion most closely related to $G_\sharp \overline{S}$ is that of a π-weak distributive law of $G_\sharp 1$ over S (Definition 5.2.3). These are more general than the weak distributive laws of Garner. It follows from Böhm's work that, subject to the splitting of certain idempotents, π-weak distributive laws of T over S correspond to certain weak monad lifts of S along U^T. We give an alternative characterisation of them in terms of the 'weak composites monads' they induce in the spirit of (i) above (Theorem 5.3.2). This is then refined in Theorem 5.3.4 to also give an analogous characterisation of weak distributive laws.

This is then applied in Section 5.4 to study when the pushforward of a lifted monad is the weak composite given by a π-weak distributive law (Theorem 5.4.4). In Section 6.1.1, we use this result to recover Garner's weak distributive law of the ultrafilter monad over the powerset monad on Set [28, §5.1].

Examples of pushforward monads No formal theory is complete without a collection of examples illustrating its relevance. Chapter 6 does this for the theory of pushforward monads developed in the previous two chapters.

The first source of examples is given by monads on FinSet and their pushforward along $J \colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$. In Section 6.1, we compute the pushforward along J and the corresponding category of algebras of several natural families of monads. Recall that the codensity monad of J is the ultrafilter monad. Many of these pushforwards along J turn out to be composite monads induced by unique distributive laws under the ultrafilter monad. This is the case, e.g. for the exception monad $- + E$ for a finite set E (Example 6.1.8) and the free M-set monad for a finite monoid M (Example 6.1.9).

The case of the powerset monad is particularly interesting. We prove that its pushforward along J is the filter monad (Theorem 6.1.14), and that it is the weak composite of a π-weak distributive law of the ultrafilter monad over the powerset monad (Theorem 6.1.18). This also allows us to show that there is no distributive law of the ultrafilter monad over the finite powerset monad (Proposition 6.1.19).

Finally, we prove that, with two trivial exceptions, the pushforward of a monad along J is never finitary. In fact, it has no rank (Theorem 6.1.24).

In Section 6.2, we consider an example of a pushforward along the inclusion $\underline{J}\colon \mathsf{FinFam}(\mathcal{E}) \to \mathsf{Fam}(\mathcal{E})$ of the free finite coproduct completion of a category $\mathcal{E}$ into its free coproduct completion. Leinster [45, Thm. 8.5] showed that the codensity monad of $\underline{J}$ is the ultracoproduct monad. When $\mathcal{E} = (\mathsf{Set}^T)^{\mathrm{op}}$ for some finitary monad T, this gives the ultraproduct construction of model theory. We give a formula for the pushforwards of monads lifted along a (cloven) Grothendieck fibration (Proposition 6.2.2). The particular example we study is that of a lift of the powerset monad along the fibration $\mathsf{FinFam}(\mathcal{E}) \to \mathsf{FinSet}$ for a category $\mathcal{E}$ with coproducts and cofiltered limits. The resulting pushforward along $\underline{J}$ is then a lift of the filter monad on Set; it is the reduced coproduct monad (Theorem 6.2.5). When $\mathcal{E} = (\mathsf{Set}^T)^{\mathrm{op}}$ as above, it gives the reduced product construction of Frayne, Morel and Scott [27].

Section 6.3 gives an explicit proof that the category of algebras of the codensity monad of the inclusion $\mathsf{Field} \hookrightarrow \mathsf{CRing}$ is $\mathsf{Prod}(\mathsf{Field})$, i.e. the free product completion of the category of fields. Although this was already derived from results by Diers in Example 4.3.23(i), we believe that the explicit calculation illustrates a general pattern in the theory of codensity monads and their relation to limit completions. We briefly consider the operations of the corresponding monad on Set, and show that it has no rank.

Lastly, Section 6.4 studies the special case of pushing forward a monad along another monad. Since the complexity of calculating Ran_G greatly increases with the complexity of G, we only provide examples of the most basic nature. In particular, we compute the codensity monad and the corresponding category of algebras of the exception monads $- + E$ on Set, where E is a set. When $|E| > 1$, the corresponding category of algebras is equivalent to $\mathsf{Prod}(\mathsf{Set})$ (Theorem 6.4.3). The case of $|E| = 1$ is special and is treated in Theorem 6.4.10.

Some directions for future work Although this thesis develops both the general theory and specific examples of pushforward monads, there is much left to be understood about them. We now list some possible directions for future work.

- There are many examples of closed bicategories (i.e. bicategories with all right extensions and lifts), such as $\mathsf{Span}(\mathcal{C})$, $\mathcal{V}$-Mat, and $\mathcal{V}$-Prof for suitable choices of $\mathcal{C}$ and $\mathcal{V}$. What does the pushforward process look like in each of these? Does it recover a well-known construction?

- Left oplax Kan extensions of 2-functors have recently been used by Tarantino and Wrigley [60] to unrelativise relative pseudomonads. This process is similar to the one we discuss in Remark 4.3.16 below. Is there a lax pushforward process for pseudomonads using right lax Kan extensions?

- As discussed at the end of the introduction to Chapter 5, one could in principle talk about pushforward monads in the category $\mathsf{MND}(\mathcal{K})$ of monads in a 2-category $\mathcal{K}$, i.e. pushforward distributive laws. In practice, this is complicated by the lack of a convenient calculus of extensions in $\mathsf{MND}(\mathcal{K})$. One approach is to consider the locally fully faithful 2-functor $\mathsf{MND}(\mathcal{K}) \hookrightarrow \mathcal{K}^{\to}$ into the arrow 2-category of $\mathcal{K}$, which reflects all extensions. When do right extensions in $\mathsf{MND}(\mathcal{K})$ or $\mathcal{K}^{\to}$ exist? What interesting pushforward distributive laws can be obtained from this construction?

- In Subsection 4.3.3 we review some results by Diers relating multimonadic functors and codensity monads. Theorem 4.3.22 says that the product-preserving functor $\widehat{G} \colon \mathsf{Prod}(\mathcal{A}) \to \mathsf{Set}$ induced by $G \colon \mathcal{A} \to \mathsf{Set}$ is monadic iff $\mathsf{Prod}(G) \colon \mathsf{Prod}(A) \to \mathsf{Prod}(\mathsf{Set})$ is monadic and the only cones in $\mathcal{A}$ sent by G to product cones in Set are the trivial ones. Is the analogous statement for other (sound) limit doctrines true? What happens if one replaces Set by a category with the relevant limits?

Chapter 2

Background

This chapter introduces all the necessary notions and notation needed for the rest of this thesis. Familiarity with basic category theory is assumed, as covered by the first three chapters of Borceux's handbook [15], as well as some notions of enriched category theory. In particular, we use Kelly's notation for weighted limits and colimits [33, Ch. 3].

The content presented here is collected from a number of sources. Section 2.1 serves as an introduction to the theory of 2-categories. Many examples are taken from Lack's helpful guide to the subject [39]. Section 2.2 focuses on the construction of extensions in a 2-category in general, and more concretely on Kan extensions of $\mathcal{V}$-functors. We have found that Wood's proarrow approach pioneered in [64] gives a gentle 2-categorical account of the latter without involving much enriched category theory. Section 2.3 summarises the main results of Street's formal theory of monads [58], which are crucial to all but one chapter of this thesis. Lastly, Section 2.4 reviews some classical theorems about monads in CAT and on Set, and introduces several monads which we use as examples.

2.1 Some 2-category theory

Categories, functors and natural transformations are the main object of study of category theory. Categories and functors assemble themselves into a (large) category CAT, but this structure alone ignores the natural transformations. Given two categories $\mathcal{A}$ and $\mathcal{B}$, the collection of functors $\mathcal{A} \to \mathcal{B}$ forms itself a category, denoted by $[\mathcal{A}, \mathcal{B}]$. Not only this, but functor composition is a functor

$$[\mathcal{B}, \mathcal{C}] \times [\mathcal{A}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{C}].$$

This allows us to view CAT as CAT-enriched category; this structure is called a 2-category.

Definition 2.1.1. A (strict) **2-category** $\mathcal{K}$ consists of

- a collection of 0-cells or objects;
- for 0-cells x and y, a category $\mathcal{K}(x, y)$, whose objects are called 1-cells and whose morphisms are called 2-cells;

- for 0-cells x, y and z, a functor

$$\mathcal{K}(y,z) \times \mathcal{K}(x,y) \xrightarrow{\circ} \mathcal{K}(x,z)$$

  called 1-cell composition;

- for each 0-cell x, an identity 1-cell in $\mathcal{K}(x,x)$, denoted by 1_x or x;

such that composition of 1-cells is strictly associative and unital, i.e.

$$(f \circ g) \circ h = f \circ (g \circ h) \qquad \text{and} \qquad 1 \circ f = f = f \circ 1.$$

For 0-cells x and y in $\mathcal{K}$, we write $f \colon x \to y$ to mean $f \in \operatorname{ob} \mathcal{K}(x,y)$. We will typically use Latin letters as names for 0- and 1-cells and Greek letters for 2-cells.

There are two basic ways to compose 2-cells in $\mathcal{K}$. If $f, g, h \colon x \to y$ and we have 2-cells $\alpha \colon f \to g$ and $\beta \colon g \to h$, then composition in the category $\mathcal{K}(x,y)$ gives a 2-cell $\beta \cdot \alpha$. This is the **vertical composite** of α and β. If instead $f, g \colon x \to y$ and $k, l \colon y \to z$, and we have 2-cells $\alpha \colon f \to g$ and $\gamma \colon k \to l$ then the action of 1-cell composition on morphisms gives a 2-cell $\gamma \circ \alpha \colon k \circ f \to l \circ g$. This is the **horizontal composite** of α and γ. We will often omit the $\circ$ symbol and simply write $\gamma\alpha \colon kf \to lg$. A frequent example of horizontal composition occurs when either γ or α is an identity. For example, we write the identity 2-cell on f as 1_f or simply f, and obtain the 2-cell $\gamma f \colon kf \to lf$. This is commonly called the **whiskering** of γ and f. Since composition of 1-cells is a functor, it preserves vertical composition of 2-cells, so that

$$l\alpha \cdot \gamma f = \gamma\alpha = \gamma g \cdot k\alpha. \tag{$*$}$$

This identity is usually called the **interchange law**.

A common way to represent 2-cells in a 2-category is using pasting diagrams. For example, the 2-cell α above can be drawn as:

$$\begin{tikzcd} x \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\alpha"'] & y \end{tikzcd}$$

The axioms of a 2-category ensure that any sensible combination of these diagrams unambiguously specifies one composite 2-cell. For example,

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g"'] \arrow[rr, bend left=60, "o"{name=O}] \arrow[dr, phantom, "\overset{\alpha}{\Swarrow}"] & y \arrow[r, "l"] \arrow[d, "h"] \arrow[dr, phantom, "\overset{\beta}{\Swarrow}"] \arrow[from=O, Rightarrow, "\gamma"'] & u \arrow[d, "m"] \\ z \arrow[r, "k"'] & w \arrow[r, "n"'] & v \end{tikzcd}$$

represents the composite $n\alpha \cdot \beta f \cdot m\gamma$.

Example 2.1.2. The primary example of a 2-category is CAT, the 2-category of locally small categories, functors and natural transformations. More generally, if $\mathcal{V}$ is a monoidal category, one obtains a 2-category $\mathcal{V}$-CAT of $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations as described by Kelly [33, §1.2].

Example 2.1.3. Any 2-category $\mathcal{K}$ gives rise to three more 2-categories by reversing the direction of the cells in different levels:

- The **horizontal dual** $\mathcal{K}^{\mathrm{op}}$ is obtained by reversing the 1-cells, so that $\mathcal{K}^{\mathrm{op}}(x,y) = \mathcal{K}(y,x)$.
- The **vertical dual** $\mathcal{K}^{\mathrm{co}}$ is obtained by reversing the 2-cells, so that $\mathcal{K}^{\mathrm{co}}(x,y) = \mathcal{K}(x,y)^{\mathrm{op}}$.
- The **horizontal and vertical dual** $\mathcal{K}^{\mathrm{coop}}$ is obtained by reversing both 1- and 2-cells, so that $\mathcal{K}^{\mathrm{coop}}(x,y) = \mathcal{K}(y,x)^{\mathrm{op}}$.

The notion of a 2-category is often too strict to accommodate many important examples. The problem arises from the requirement that composition of 1-cells be strictly associative and unital. The more general notion is that of a **bicategory**, where the equations at the end of Definition 2.1.1 are replaced by natural isomorphisms

$$(f \circ g) \circ h \xrightarrow{\sim} f \circ (g \circ h) \qquad \text{and} \qquad 1 \circ f \xrightarrow{\sim} f \xleftarrow{\sim} f \circ 1$$

subject to two coherence conditions (see Bénabou [12, §1.1]).

Example 2.1.4. A monoidal category amounts to the data of a one-object bicategory. Indeed, if $\mathcal{V}$ is a monoidal category, there is a bicategory $\mathrm{B}\mathcal{V}$ with a single 0-cell $\bullet$ and $\mathrm{B}\mathcal{V}(\bullet,\bullet) = \mathcal{V}$. Composition of 1-cells corresponds to the monoidal product in $\mathcal{V}$, and the identity 1-cell is the monoidal unit.

Example 2.1.5. There is a bicategory $\mathcal{R}$ where 0-cells are (not-necessarily-commutative, unital) rings, 1-cells $A \to B$ are (B,A)-bimodules and 2-cells $M \to N$ are bimodule homomorphisms. The composite of the (B,A)-bimodule M with the (C,B)-bimodule N is the (C,A)-bimodule given by the tensor product $N \otimes_B M$. The identity bimodule on the ring A is A, seen as a bimodule over itself.

Note that $\mathcal{R}$ is not a 2-category: the tensor product of bimodules is not strictly associative or unital. Instead, we have canonical isomorphism $(P \otimes_C N) \otimes_B M \cong P \otimes_C (N \otimes_B M)$ and similarly for tensoring with the identities.

Example 2.1.6. One can vastly generalise the previous example as follows. Let $\mathcal{V}$ be a cocomplete monoidal category where $- \otimes -$ preserves colimits in each variable. There is a bicategory $\mathcal{V}$-Prof introduced by Bénabou [13, §3], where 0-cells are small $\mathcal{V}$-categories. A 1-cell $\Phi \colon \mathcal{C} \to \mathcal{D}$ is a **$\mathcal{V}$-profunctor** (or distributor, or bimodule). We will use the notation $\mathcal{C} \nrightarrow \mathcal{D}$ to distinguish profunctors from functors. Such a profunctor is a function $\Phi\colon \operatorname{ob}\mathcal{D} \times \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{V}$ with an action of $\mathcal{C}$ on the left and of $\mathcal{D}$ on the right, i.e. families of morphisms

$$\mathcal{C}(c,c') \otimes \Phi(d,c) \longrightarrow \Phi(d,c') \qquad \text{and} \qquad \Phi(d,c) \otimes \mathcal{D}(d',d) \longrightarrow \Phi(d',c)$$

compatible with the composition and identities of $\mathcal{D}$ and $\mathcal{C}$. A 2-cell $\alpha\colon \Phi \to \Psi$ is a family of morphisms $\alpha_{d,c}\colon \Phi(d,c) \to \Psi(d,c)$ in $\mathcal{V}$ suitably compatible with the actions.

Given profunctors $\Phi\colon \mathcal{C} \to \mathcal{D}$ and $\Psi\colon \mathcal{D} \to \mathcal{E}$, their composite is computed using a coend in $\mathcal{V}$:

$$(\Psi \otimes \Phi)(e,c) = \int^{d \in \mathcal{D}} \Phi(d,c) \otimes \Psi(e,d).$$

The identity profunctor on $\mathcal{C}$ is simply $\mathcal{C}(-,-)$. The associator is given by the preservation of colimits of $\otimes$, and the unitors correspond to the density formula.

The most common situation in which we consider $\mathcal{V}$-Prof is when $\mathcal{V}$ is symmetric monoidal closed. In this case, a profunctor $\mathcal{C} \to \mathcal{D}$ amounts to a $\mathcal{V}$-functor $\mathcal{D}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$, and a 2-cell $\Phi \to \Psi$ is a $\mathcal{V}$-natural transformation.

The bicategory $\mathcal{R}$ of the previous example is the full sub-bicategory of Ab-Prof spanned by the one-object Ab-categories.

There are several notions of maps between bicategories. The most general one is that of a **lax functor** $F\colon \mathcal{K} \to \mathcal{L}$. It amounts to an assignment of

- a 0-cell Fx of $\mathcal{L}$ for each 0-cell x of $\mathcal{K}$,
- a functor $F\colon \mathcal{K}(x,y) \to \mathcal{L}(Fx,Fy)$ for each pair x,y of 0-cells of $\mathcal{K}$,
- comparison 2-cells $Ff \circ Fg \to F(f \circ g)$ and $1_{Fx} \to F(1_x)$ for each 0-cell x of $\mathcal{K}$ and pair f,g of composable 1-cells of $\mathcal{K}$,

subject to coherence conditions analogous to those of a lax monoidal functor between monoidal categories. If the comparison maps are invertible, we say that F is a **pseudofunctor**. If, moreover, $\mathcal{K}$ and $\mathcal{L}$ are 2-categories and the comparison maps are identities, then we say that F is a **2-functor**. If P is a property of functors, we say that a lax functor $F\colon \mathcal{K} \to \mathcal{L}$ is **locally** P if the functors $\mathcal{K}(x,y) \to \mathcal{L}(Fx,Fy)$ are P.

Example 2.1.7. If $\mathcal{V}$ is a symmetric monoidal category, then every $\mathcal{V}$-category $\mathcal{C}$ has an opposite $\mathcal{C}^{\mathrm{op}}$. This extends to a 2-functor $(-)^{\mathrm{op}}\colon \mathcal{V}\text{-CAT}^{\mathrm{co}} \to \mathcal{V}\text{-CAT}$ which is its own inverse.

One of the main uses of 2-categories, or bicategories more generally, is to view them as settings for formal category theory. Many notions in category theory, such as equivalences, adjunctions and Kan extensions, can be defined in a bicategory in such a way that one recovers the classical notions when taking the bicategory in question to be CAT. For example, if $f\colon x \to y$ and $g\colon y \to x$ are 1-cells in a 2-category, an **adjunction** $f \dashv g$ amounts to two 2-cells $\eta\colon x \to gf$ and $\epsilon\colon fg \to y$, called the unit and the counit, satisfying the familiar triangle identities. Since this is expressed purely in terms of composition and identities, every pseudofunctor sends adjoint 1-cells to adjoint 1-cells.

Example 2.1.8. Let $\mathcal{V}$ be as in Example 2.1.6. There is an identity-on-objects pseudofunctor $(-)_*\colon \mathcal{V}\text{-Cat} \to \mathcal{V}\text{-Prof}$ which sends a $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ to the profunctor F_* given by

$$F_*(d,c) = \mathcal{D}(d,Fc),$$

and a $\mathcal{V}$-natural transformation $\alpha\colon F \to G$ to the $\mathrm{ob}\,\mathcal{C}$-indexed family of maps

$$\mathcal{D}(d, \alpha_c)\colon \mathcal{D}(d, Fc) \longrightarrow \mathcal{D}(d, Gc).$$

The Yoneda lemma implies that $(-)_*$ is locally fully faithful.

There is also an identity-on-objects pseudofunctor $(-)^*\colon \mathcal{V}\text{-}\mathsf{Cat}^{\mathrm{coop}} \to \mathcal{V}\text{-}\mathsf{Prof}$ sending F to the profunctor F^* given by

$$F^*(c, d) = \mathcal{D}(Fc, d),$$

and α to

$$\mathcal{D}(\alpha_c, d)\colon \mathcal{D}(Gc, d) \longrightarrow \mathcal{D}(Fc, d).$$

Again, $(-)^*$ is locally fully faithful.

Note that the density formula implies that, given a string of two functors and a profunctor

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \overset{\Phi}{\longrightarrow\!\!\!\!\!\!\!+\;\;} \mathcal{C} \xleftarrow{G} \mathcal{D},$$

one has

$$(\Phi \otimes F_*)(c, a) = \int^{b \in \mathcal{B}} \mathcal{B}(b, Fa) \otimes \Phi(c, b) \cong \Phi(c, Fa), \tag{2.1}$$

$$(G^* \otimes \Phi)(d, b) = \int^{c \in \mathcal{C}} \Phi(c, b) \otimes \mathcal{C}(Gd, c) \cong \Phi(Gd, b). \tag{2.2}$$

The two profunctors obtained from a functor $F\colon \mathcal{C} \to \mathcal{D}$ in this way are always adjoint to each other: $F_* \dashv F^*$. The unit of this adjunction is given by the action of F on morphisms:

$$\eta_{c,c'}\colon \mathcal{C}(c, c') \xrightarrow{F_{c,c'}} \mathcal{D}(Fc, Fc') \cong (F^* \otimes F_*)(c, c'). \tag{2.3}$$

The counit is the morphism

$$\epsilon_{d,d'}\colon (F_* \otimes F^*)(d, d') \cong \int^{c \in \mathcal{C}} \mathcal{D}(Fc, d') \otimes \mathcal{D}(d, Fc) \longrightarrow \mathcal{D}(d, d') \tag{2.4}$$

induced by the cowedge whose legs are given by composition in $\mathcal{D}$.

The pseudofunctor $(-)_*\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$ is an example of a **proarrow equipment** in the sense of Wood [64, §1], i.e. a locally fully faithful identity-on-objects pseudofunctor which sends every 1-cell to a left adjoint and whose codomain has all right lifts and extensions (see Definition 2.2.1 and Example 2.2.5). This extra structure allows one to develop a more expressive formal category than when using a mere bicategory. For example, one can talk about pointwise extensions in a proarrow equipment.

Much of our theory will take place inside a fixed bicategory. For such purposes, it is common and convenient to take this bicategory to actually be a 2-category. This does not constitute any loss of generality: for any bicategory $\mathcal{K}$ there is a 2-category $\mathrm{s}\mathcal{K}$ and a pseudofunctor $F\colon \mathcal{K} \to \mathrm{s}\mathcal{K}$ which is a **biequivalence**, i.e. each of the functors $\mathcal{K}(x, y) \to \mathrm{s}\mathcal{K}(Fx, Fy)$ is an equivalence and F is surjective on 0-cells up to equivalence in s$\mathcal{K}$. This is known as the coherence theorem for bicategories. For a streamlined treatment, we direct the reader to Leinster's notes [44].

2.2 Extensions and (co)density

The construction that underlies most of the content of this thesis is that of an extension in a 2-category $\mathcal{K}$, which we review in this section. We pay special attention to the special case of pointwise extensions in $\mathcal{V}$-CAT, where $\mathcal{V}$ is a **Bénabou cosmos**, i.e. a complete and cocomplete symmetric monoidal closed category.

If $g\colon x \to y$ is a 1-cell and z is a 0-cell, let $g_\circ\colon \mathcal{K}(z,x) \to \mathcal{K}(z,y)$ and $g^\circ\colon \mathcal{K}(y,z) \to \mathcal{K}(x,z)$ denote the functors given by composing with g on the left and right, respectively. If $f\colon x \to z$, we write $g^\circ{\downarrow}f$ for the comma category whose objects are pairs (k,α) of a 1-cell $k\colon y \to z$ and a 2-cell $\alpha\colon kg \to f$.

Definition 2.2.1. Given 1-cells $f\colon x \to z$ and $g\colon x \to y$ in $\mathcal{K}$, a **right extension** of f along g is a terminal object of $g^\circ{\downarrow}f$.

In simple terms, it is a pair $(\operatorname{ran}_g f, \epsilon)$ of a 1-cell $\operatorname{ran}_g f\colon y \to z$ and a 2-cell $\epsilon\colon (\operatorname{ran}_g f)g \to f$ such that, for all pairs (k,α) fitting into the left-hand side diagram, there exists a unique 2-cell $\widehat{\alpha}$ fitting into the right-hand side one:

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[dr, "g"'] & \arrow[d, phantom, "\alpha\Uparrow"] & z & & x \arrow[rr, "f"] \arrow[dr, "g"'] & \arrow[d, phantom, "\epsilon\Uparrow"] & z. \\
& y \arrow[ur, "k"'] & & = & & y \arrow[ur, "\operatorname{ran}_g f"'{name=R}] \arrow[ur, bend right=70, "k"'{name=K}] \arrow[Rightarrow, from=K, to=R, "\widehat{\alpha}"'] &
\end{tikzcd}$$

The 2-cell ϵ is called the **counit** of the right extension. A **left extension**, **right lift** and **left lift** are a right extension in $\mathcal{K}^{\mathrm{co}}$, $\mathcal{K}^{\mathrm{op}}$ and $\mathcal{K}^{\mathrm{coop}}$, respectively. These are denoted by lan, rift and lift, respectively. Being defined by a universal property, right/left extensions/lift are unique up to unique isomorphism.

Another way to view right extensions is to note that $\operatorname{ran}_g f$ is the value of at f of a (partial) right adjoint ran_g to g°. All right extensions along g exists iff this partial adjoint is total. In particular, $\operatorname{ran}_g\colon \mathcal{K}(x,z) \to \mathcal{K}(y,z)$ is a partial functor, so that if $\operatorname{ran}_g f$ and $\operatorname{ran}_g f'$ exist and $\alpha\colon f \to f'$ is a 2-cell, there is a 2-cell $\operatorname{ran}_g \alpha\colon \operatorname{ran}_g f \to \operatorname{ran}_g f'$. Dually, left extension along g provides a (partial) left adjoint to g°, and right/left lifts along g provide (partial) right/left adjoints to $g_\circ$.

Much like limits in a category, we can ask when right extensions are preserved. Given the extra structure in a 2-category, there are now two situations in which we talk about preservation: internally and externally.

Definition 2.2.2. Consider the diagram in $\mathcal{K}$:

$$\begin{tikzcd}
x \arrow[rr, "f"] \arrow[dr, "g"'] & & z \arrow[r, "h"] & w. \\
& y \arrow[ur, "k"'] \arrow[u, phantom, "\epsilon\Uparrow"] & &
\end{tikzcd}$$

The 1-**cell** h **preserves the right extension** of f along g if

$$\begin{aligned}
&(k, \epsilon) \text{ is a right extension of } f \text{ along } g \\
&\implies (hk, h\epsilon) \text{ is a right extension of } hf \text{ along } g.
\end{aligned}$$

A right extension is **absolute** if it is preserved by every 1-cell.

A **pseudofunctor** $F\colon \mathcal{K} \to \mathcal{L}$ **preserves the right extension** of f along g if

$$\begin{aligned}
&(k, \epsilon) \text{ is a right extension of } f \text{ along } g \\
&\implies (Fk, F\epsilon) \text{ is a right extension of } Ff \text{ along } Fg.
\end{aligned}$$

If instead the converse is true, we say that F **reflects** this right extension.

We now collect some well-known properties of extensions (and dually lifts) which we will find useful. Their straightforward proofs are omitted.

Lemma 2.2.3. *The following statements hold:*

(i) *Locally fully faithful pseudofunctors reflect extensions.*

(ii) *Right adjoint* 1*-cells preserve right extensions.*

(iii) *If $f\colon y \to x$, $g\colon x \to y$ and $h\colon x \to z$ are* 1*-cells then and $f \dashv g$ with counit $\epsilon\colon fg \to x$, then $(hf, h\epsilon)$ is an absolute right extension of h along g.*

(iv) *In the diagram*

$$\begin{tikzcd}
& w & \\
x \arrow[ur, "f"] \arrow[r, "g"'] & y \arrow[u, "k"] \arrow[ul, phantom, "\overset{\Nwarrow}{\epsilon}"] & z, \arrow[ul, "l"'] \arrow[l, "h"] \arrow[ul, phantom, shift right=3, "\overset{\Nwarrow}{\alpha}"]
\end{tikzcd}$$

if (k, ϵ) is a right extension, then $(l, \epsilon \cdot \alpha g)$ is a right extension iff (l, α) is.

Example 2.2.4. Let $\mathcal{V}$ be a Bénabou cosmos. There are three successively stronger notions of right extension for $\mathcal{V}$-functors. Let $F\colon \mathcal{A} \to \mathcal{C}$, $G\colon \mathcal{A} \to \mathcal{B}$ and $H\colon \mathcal{B} \to \mathcal{C}$ be $\mathcal{V}$-functors and $\epsilon\colon HG \to F$ be a $\mathcal{V}$-natural transformation.

(i) The weakest notion is asking that (H, ϵ) be a right extension of F along G in the 2-category $\mathcal{V}$-CAT.

(ii) The next is that of a right extension in the $\mathcal{V}$-CAT-category of $\mathcal{V}$-categories. While a right extension in a 2-category amounts to a bijection between sets of 2-cells, this one asserts that the composite

$$[\mathcal{B},\mathcal{C}](K,H) \xrightarrow{-G} [\mathcal{A},\mathcal{C}](KG,HG) \xrightarrow{\epsilon_*} [\mathcal{A},\mathcal{C}](KG,F)$$

is an isomorphism between the $\mathcal{V}$-objects of natural transformations for all $K\colon \mathcal{B} \to \mathcal{C}$. These two notions coincide when the functor $\mathcal{V}(I,-)\colon \mathcal{V} \to \mathsf{Set}$ reflects isomorphisms.

(iii) The third and strongest notion is that of a $\mathcal{V}$-enriched right Kan extension of F along G, as described by Kelly [33, Ch. 4]. It asserts that for each $b \in \mathcal{B}$ the cylinder

$$\mathcal{B}(b,G-) \xrightarrow{H} \mathcal{C}(Hb,HG-) \xrightarrow{\epsilon_*} \mathcal{C}(Hb,F-)$$

realise Hb as the weighted limit $\{\mathcal{B}(b,G-),F-\}$ in $\mathcal{C}$. If this limit exists for each $b \in \mathcal{B}$, then they assemble into a $\mathcal{V}$-functor which is the $\mathcal{V}$-enriched right Kan extension of F along G. The required limits clearly exist when $\mathcal{A}$ is small and $\mathcal{C}$ is complete. Extensions in the 2-category $\mathcal{V}$-CAT obtained this way are called **pointwise** and denoted by $\mathrm{Ran}_G\, F$ with a capital R.

When $\mathcal{V} = \mathsf{Set}$, this weighted limit can be further written as a conical one:

$$(\mathrm{Ran}_G\, F)(b) = \lim \left(b{\downarrow}G \xrightarrow{\Pi_b} \mathcal{A} \xrightarrow{F} \mathcal{C} \right), \tag{2.5}$$

where Π_b is the forgetful functor from the comma category.

Dually, $\mathcal{V}$-enriched left Kan extensions (or pointwise left extensions in $\mathcal{V}$-CAT) are given by a weighted colimit formula:

$$(\mathrm{Lan}_G\, F)(b) = \mathcal{B}(G-,b) \star F- \,. \tag{2.6}$$

Example 2.2.5. With $\mathcal{V}$ as above, we now consider extensions in $\mathcal{K} = \mathcal{V}$-Prof. Given $\mathcal{V}$-profunctors $\Psi\colon \mathcal{A} \nrightarrow \mathcal{B}$ and $\Phi\colon \mathcal{A} \nrightarrow \mathcal{C}$, the right extension of Φ along Ψ always exists, and is given by the weighted limit

$$(\mathrm{ran}_\Psi\, \Phi)(c,b) = \{\Psi(b,-),\Phi(c,-)\}$$

in $\mathcal{V}$ for $c \in \mathcal{C}$ and $b \in \mathcal{B}$. This is precisely the $\mathcal{V}$-object of natural transformations between the functors $\Psi(b,-)$ and $\Psi(c,-)\colon \mathcal{A} \to \mathcal{V}$.

Dually, $\mathcal{V}$-Prof also has right lifts. If $\Theta\colon \mathcal{C} \nrightarrow \mathcal{B}$, then

$$(\mathrm{rift}_\Psi\, \Theta)(a,c) = \{\Psi(-,a),\Theta(-,c)\} \tag{2.7}$$

for $a \in \mathcal{A}$ and $c \in \mathcal{C}$. Again, this is the $\mathcal{V}$-object of natural transformations from $\Psi(-,a)$ to $\Theta(-,c)\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$.

Let $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{A} \to \mathcal{B}$ be $\mathcal{V}$-functors with small domain and codomain.

If $\operatorname{Ran}_G F$ exists, then

$$\begin{aligned}(\operatorname{ran}_{G_*} F_*)(b,c) &= \{\mathcal{B}(b,G-),\mathcal{C}(c,F-)\}\\ &\cong \mathcal{C}(c,\{\mathcal{B}(b,G-),F-\})\\ &= \mathcal{C}(c,(\operatorname{Ran}_G F)(b))\\ &= (\operatorname{Ran}_G F)_*(b,c)\end{aligned}$$

because representables preserve limits. This means that the pseudofunctor $(-)_*\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$ preserves those right extensions in $\mathcal{V}$-Cat that are pointwise. Conversely, if $\operatorname{ran}_G F$ exists and is preserved by $(-)_*$, then

$$\begin{aligned}\mathcal{C}(c,(\operatorname{ran}_G F)(b)) &= (\operatorname{ran}_G F)_*(b,c)\\ &\cong (\operatorname{ran}_{G_*} F_*)(b,c)\\ &= \{\mathcal{B}(b,G-),\mathcal{C}(c,F-)\}\\ &\cong \mathcal{C}(c,\{\mathcal{B}(b,G-),F-\}),\end{aligned}$$

and so $\operatorname{ran}_G F$ is pointwise by the Yoneda lemma. Thus, a right extension in $\mathcal{V}$-Cat is pointwise iff it is preserved by the pseudofunctor $(-)_*$. Dually, pointwise left extensions are those sent by $(-)^*\colon \mathcal{V}\text{-}\mathsf{Cat}^{\mathrm{coop}} \to \mathcal{V}\text{-}\mathsf{Prof}$ to right lifts.

Remark 2.2.6. Since $(-)_*$ reflects extensions by Lemma 2.2.3(i) because it is locally fully faithful, this confirms that pointwise extensions are in particular extensions in $\mathcal{V}$-Cat. Moreover, since we have characterised them purely in terms of 2-categorical language, it is easy to see that the last three statements of Lemma 2.2.3 hold for $\mathcal{V}$-Cat when the words ‘right extension’ are replaced by ‘pointwise right extension’.

The use of the word extension can be misleading, since in general the (co)unit of an extension need not an isomorphism. In $\mathcal{V}$-CAT there is an often-used way to ensure that a pointwise extension is a genuine extension.

Lemma 2.2.7. *If G is a fully faithful $\mathcal{V}$-functor, then the counit of any pointwise right Kan extension along it is invertible.*

Once again we can use the proarrow equipment $(-)_*\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$ to verify this without much knowledge about enriched category theory. First, we internalise another basic notion of category theory to a 2-category.

Definition 2.2.8. A 1-cell $f\colon x \to y$ in a 2-category $\mathcal{K}$ is **fully faithful** if the functor $f_\circ\colon \mathcal{K}(z,x) \to \mathcal{K}(z,y)$ is fully faithful for all 0-cells z.

Lemma 2.2.9. *A left adjoint 1-cell $f\colon x \to y$ is fully faithful iff the unit of the adjunction is invertible.*

Proof. This is famously true in CAT (see e.g. Borceux’s book [15, Vol. I, Prop. 3.4.1]). For each 0-cell z, there is a representable 2-functor $\mathcal{K}(z,-)\colon \mathcal{K} \to \mathsf{CAT}$ which sends f to $f_\circ$. Since 2-functors preserve adjunctions, if $f \dashv g$ is an adjunction with unit $\eta\colon x \to gf$, then $f_\circ \dashv g_\circ$ is an adjunction with unit $\eta \cdot -$. Then

$$\begin{aligned} f \text{ is fully faithful } &\iff \forall z,\ f_\circ\colon \mathcal{K}(z,x) \to \mathcal{K}(z,y) \text{ is fully faithful} \\ &\iff \forall z,\ \eta -\colon 1_{\mathcal{K}(z,x)} \to g_\circ f_\circ \text{ is invertible} \\ &\iff \eta \text{ is invertible,} \end{aligned}$$

where the last forward implication is seen by taking $z = x$. ☐

Lemma 2.2.10. *A $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ is fully faithful iff F_* is a fully faithful 1-cell of $\mathcal{V}$-*Prof*. If so, F is also a fully faithful 1-cell of $\mathcal{V}$-*Cat*.*

Proof. By the description of the unit of $F_* \dashv F^*$ in (2.3) and the previous lemma, it is clear that F_* is fully faithful iff F is fully faithful as a $\mathcal{V}$-functor. Since any locally fully faithful pseudofunctor reflects fully faithful 1-cells, it follows that F is fully faithful as a 1-cell of $\mathcal{V}$-Cat. ☐

Note that a fully faithful 1-cell of $\mathcal{V}$-Cat need not be fully faithful as a $\mathcal{V}$-functor. This is true however if $\mathcal{V}(I,-)\colon \mathcal{V} \to$ Set reflects isomorphisms. It is also true if F has a right adjoint by Lemma 2.2.9, since then the unit is an invertible 2-cell of $\mathcal{V}$-Cat which is sent to an invertible 2-cell by $(-)_*$.

Proof of Lemma 2.2.7. Let $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{A} \to \mathcal{B}$ be $\mathcal{V}$-functors such that $\mathrm{Ran}_G F$ exists and G is fully faithful. Since the extension is pointwise, we have $((\mathrm{Ran}_G F)G)_* \cong (\mathrm{ran}_{G_*} F_*) \otimes G_*$. Since $G_* \dashv G^*$, Lemma 2.2.3(iii) implies that this is isomorphic to $\mathrm{ran}_{G^*}(\mathrm{ran}_{G_*} F_*)$. Part (iv) of the same lemma further simplifies this to $\mathrm{ran}_{G^* \otimes G_*} F_*$. Since G_* is fully faithful by assumption, Lemma 2.2.9 gives that $G^* \otimes G_*$ is the identity profunctor, so this is isomorphic to F_*. Since $(-)_*$ is locally fully faithful, this implies that $(\mathrm{Ran}_G F)G \cong F$. ☐

The pointwise left and right extensions of a $\mathcal{V}$-functor along itself are an important special case.

Definition 2.2.11. A $\mathcal{V}$-functor $G\colon \mathcal{A} \to \mathcal{B}$ is **dense** if $(1_\mathcal{B}, 1_G)$ is a pointwise left extension of G along itself. Dually, it is **codense** if it is a pointwise right extension.

The use of the word ‘dense’ is motivated by recalling that the pointwise left extension of G along itself is computed as a weighted colimit in $\mathcal{B}$. Hence, if G is dense then

$$b \cong \mathcal{B}(G-, b) \star G-$$

for all $b \in \mathcal{B}$, so that each object of $\mathcal{B}$ is a colimit of objects in the image of G in a canonical way. The next lemma gives a related well-known characterisation of density which is often taken as the definition.

Lemma 2.2.12. *A $\mathcal{V}$-functor $G\colon \mathcal{A} \to \mathcal{B}$ is dense iff $\mathcal{B}(G-,-)\colon \mathcal{B} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is fully faithful.*

Proof. By (2.7), G is dense iff $\{\mathcal{B}(G-,b), \mathcal{B}(G-,b')\} \cong \mathcal{B}(b,b')$ for $b, b' \in \mathcal{B}$. ☐

Dense $\mathcal{V}$-functors satisfy some convenient closure properties which we now record. They are proved in Kelly's book [33, §5.2], although a proof using the proarrow equipment $(-)_*$ is also possible.

Lemma 2.2.13. *Let $G\colon \mathcal{A} \to \mathcal{B}$ and $H\colon \mathcal{B} \to \mathcal{C}$ be $\mathcal{V}$-functors.*

(i) *If G is dense and H preserves $\operatorname{Ran}_G G$, then H is dense iff HG is.*

(ii) *If H is fully faithful and HG is dense, then both G and H are dense.*

Remark 2.2.14. Note that the condition in (i) that H preserve $\operatorname{Ran}_G G$ cannot be dropped. The following example is due to Kelly [33, §5.2]: the ordinary functors $G\colon \mathsf{1} \to \mathsf{FinSet}$ picking out 1 and $y\colon \mathsf{FinSet} \to [\mathsf{FinSet}^{\mathrm{op}}, \mathsf{Set}]$ are both dense, but their composite is not.

Dense $\mathcal{V}$-functors which are also fully faithful play an important role: they correspond to free cocompletions under classes of colimits. Note that, by part (ii) of the previous lemma, the full image factorisation of any dense $\mathcal{V}$-functor is of this type. There are two slightly different notions which one might refer to as a free cocompletion. Given a category $\mathcal{A}$, we can freely adjoin colimits for

(i) a class $\mathfrak{D}$ of diagrams, seen as pairs (W, D) where $W\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{V}$ and $D\colon \mathcal{I} \to \mathcal{A}$, to obtain a fully faithful functor $J\colon A \hookrightarrow \mathfrak{D}[\mathcal{A}]$ such that $W \star JD$ exists for each $(W, D) \in \mathfrak{D}$; or

(ii) a class $\mathfrak{C}$ of functors $W\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{V}$, thought of as the weights of the colimits being adjoined, to obtain a fully faithful functor $J\colon \mathcal{A} \hookrightarrow \mathfrak{C}(\mathcal{A})$ such that $\mathfrak{C}(\mathcal{A})$ has all colimits weighted by elements of $\mathfrak{C}$. We call such colimits **$\mathfrak{C}$-colimits**, and categories having all of them **$\mathfrak{C}$-cocomplete**.

The sense in which these cocompletions are free is subtly different in each case:

(i) The functor $J\colon \mathcal{A} \hookrightarrow \mathfrak{D}[\mathcal{A}]$ has the property that, for any functor $F\colon \mathcal{A} \to \mathcal{B}$ such that $W \star FD$ exists in $\mathcal{B}$ for each $(W, D) \in \mathfrak{D}$, there exists a unique-up-to-isomorphism functor $\widehat{F}\colon \mathfrak{D}[\mathcal{A}] \to \mathcal{B}$ such that $F \cong \widehat{F}J$ and $\widehat{F}$ preserves the colimit $W \star JD$ for each $(W, D) \in \mathfrak{D}$.

The category $\mathfrak{D}[\mathcal{A}]$ is the full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathsf{Set}]$ on the representables and the colimits $W \star yD$ for $(W, D) \in \mathfrak{D}$, where $y\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is the Yoneda embedding. The functor $J\colon \mathcal{A} \hookrightarrow \mathfrak{D}[\mathcal{A}]$ is the restriction of y. For F as above, $\widehat{F}$ is given by $\operatorname{Lan}_J F$.

(ii) The functor $J\colon \mathcal{A} \hookrightarrow \mathfrak{C}(\mathcal{A})$ has the property that, for any functor $F\colon \mathcal{A} \to \mathcal{B}$ where $\mathcal{B}$ is $\mathfrak{C}$-cocomplete, there exists a unique-up-to-isomorphism functor $\widehat{F}\colon \mathfrak{C}(\mathcal{A}) \to \mathcal{B}$ such that $F \cong \widehat{F}J$ and $\widehat{F}$ is **$\mathfrak{C}$-cocontinuous**, i.e. $\widehat{F}$ preserves all $\mathfrak{C}$-colimits.

When the domain of each weight in $\mathfrak{C}$ is small, the category $\mathfrak{C}(\mathcal{A})$ is the closure of the representables in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ under colimits whose shapes are in $\mathfrak{C}$. This closure is constructed through a transfinite inductive process. Letting $\mathcal{A}_0$ denote the full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ on the representables and α be an ordinal, we set $\mathcal{A}_{\alpha+1}$ to be the full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$

containing $\mathcal{A}_\alpha$ together with all the $\mathfrak{C}$-colimits of diagrams whose image is in $\mathcal{A}_\alpha$. For a limit ordinal β, we let $\mathcal{A}_\beta = \bigcup_{\alpha<\beta} \mathcal{A}_\alpha$. Then $\mathfrak{C}(\mathcal{A}) = \bigcup_\alpha \mathcal{A}_\alpha$. If the cardinality of the domains of the weights in $\mathfrak{C}$ is strictly bounded by some regular κ, then $\mathfrak{C}(\mathcal{A}) = \mathcal{A}_\kappa$. The functor $J \colon \mathcal{A} \to \mathfrak{C}(\mathcal{A})$ is again the restriction of y, and $\widehat{F} \cong \mathrm{Lan}_J F$.

That the given definitions of $\mathfrak{D}[\mathcal{A}]$ and $\mathfrak{C}(\mathcal{A})$ satisfy these properties is shown in Kelly's book [33, §4.10 and §5.7]. Note that (ii) is a special case of (i) by construction: given a class of weights $\mathfrak{C}$, we can take the class $\mathfrak{D}$ of pairs $(W, 1_{\mathcal{A}})$ where $W \in \mathfrak{C}(\mathcal{A})$ (or, more conveniently, small diagram pairs with the same colimit) to get $\mathfrak{D}[\mathcal{A}] = \mathfrak{C}(\mathcal{A})$. Even so, the universal property expressed by (i) in this case is stronger than the one in (ii).

The second construction is much better behaved in practice, because it is uniform across the category $\mathcal{A}$. In fact, $\mathfrak{C}(\mathcal{A})$ is the value at $\mathcal{A}$ of a lax-idempotent pseudomonad $\mathfrak{C}(-)$ on $\mathcal{V}$-CAT, in the terminology of Kelly and Lack [34].

Theorem 2.2.15. *A fully faithful functor $F \colon \mathcal{A} \to \mathcal{B}$ is dense iff it is equivalent to a free cocompletion of the form $J \colon \mathcal{A} \hookrightarrow \mathfrak{D}[\mathcal{A}]$ for some class of diagrams $\mathfrak{D}$. Moreover, letting $\mathfrak{C}_F = \{\mathcal{B}(F-, b) \mid b \in \mathcal{B}\}$, the following are equivalent:*

(i) *$\mathcal{B}$ is $\mathfrak{C}_F$-cocomplete and $\mathcal{B}(F-,-) \colon \mathcal{B} \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is $\mathfrak{C}_F$-cocontinuous;*

(ii) *F is equivalent to the free cocompletion $J \colon \mathcal{A} \hookrightarrow \mathfrak{C}_F(\mathcal{A})$;*

(iii) *F is equivalent to a free cocompletion $J \colon \mathcal{A} \hookrightarrow \mathfrak{C}(\mathcal{A})$ for some class of weights $\mathfrak{C}$.*

Proof. Any free cocompletion $J \colon \mathcal{A} \hookrightarrow \mathfrak{D}[\mathcal{A}]$ is dense by Lemma 2.2.13(ii) because the Yoneda embedding $y \colon \mathcal{A} \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is dense. Conversely, if F is dense then $\mathcal{B}$ is equivalent to a full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ containing $\mathcal{B}(F-, b) \cong \mathcal{B}(F-, b) \star y$ for each $b \in \mathcal{B}$, so that we may take $\mathfrak{D} = \{(\mathcal{B}(F-, b), y) \mid b \in \mathcal{B}\}$.

For the second part, the equivalence of (i) and (ii) follows from Proposition 5.62 of Kelly [33]. Trivially, (ii) implies (iii). For the last implication, it suffices to show that (i) holds for $F = J \colon \mathcal{A} \hookrightarrow \mathfrak{C}(\mathcal{A})$. Given $W \in \mathfrak{C}(\mathcal{A})$, note that $\mathfrak{C}(\mathcal{A})(J-, W) \cong W$ by the Yoneda lemma, so that $\mathfrak{C}(\mathcal{A})(J-,-)$ is simply the inclusion $K \colon \mathfrak{C}(\mathcal{A}) \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$.

One proves that $\mathfrak{C}(\mathcal{A})$ is $\mathfrak{C}_J$-cocomplete and that K is $\mathfrak{C}_J$-cocontinuous by transfinite induction. If $W \in \mathcal{A}_0$, then $W = ya$ for some $a \in \mathcal{A}$ and $W \star D \cong Da$ for any $D \colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$, so $\mathfrak{C}(\mathcal{A})$ has W-indexed colimits and they are preserved by K. If $W \in \mathcal{A}_{\alpha+1}$, then $W \cong X \star E$ for some $X \colon \mathcal{I}^{\mathrm{op}} \to \mathcal{V} \in \mathfrak{C}$ and $E \colon \mathcal{I} \to \mathcal{A}_\alpha$. For any $D \colon \mathcal{A} \to \mathfrak{C}(\mathcal{A})$, we have $W \star D \cong (X \star E) \star D \cong X \star (E- \star D)$ which exists and is preserved by K by induction because $Ei \in \mathcal{A}_\alpha$ for each $i \in \mathcal{I}$. The limit case is trivial. □

2.3 The formal theory of monads

Much of this thesis is concerned with monads. The definition of a monad makes sense not just on a category, but on a 0-cell of a bicategory. We work more simply with a 2-category, since the coherence theorem ensures no loss of generality. This more general notion of monad was identified by Street in [58], where he defined a 2-category of monads *in* a 2-category. We will cover some useful aspects of the classical theory of monads on categories later in Section 2.4.

For the rest of this section, fix a 2-category $\mathcal{K}$. The constructions we introduce will often involve axioms amounting to the commutativity of a diagram of 2-cells in $\mathcal{K}$. Each of these diagrams has been given a suggestive label, written in blue inside the face of the diagram. These labels will be used in later chapters to justify the commutativity of larger diagrams. They follow the syntax $(A.B)$, where A is an instance of the structure the axiom applies to and B is used to identify which axiom of the corresponding structure we mean. When a face is labelled inside a proof, the label is a hyperlink to the definition of the relevant axiom. Readers using a hard copy may refer to the index of face labels on page 149.

Definition 2.3.1. A **monad** on a 0-cell x of $\mathcal{K}$ is a triple (t, μ^t, η^t) where $t\colon x \to x$ is a 1-cell, and $\mu^t\colon t^2 \to t$ and $\eta^t\colon 1 \to t$ are 2-cells such that the following commute:

$$\begin{tikzcd}
t^3 \arrow[r, "\mu^t t"] \arrow[d, "t\mu^t"'] \arrow[dr, phantom, "(t.a)"] & t^2 \arrow[d, "\mu^t"] \\
t^2 \arrow[r, "\mu^t"'] & t,
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[r, "\eta^t t"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "(t.l)"] & t^2 \arrow[d, "\mu^t"] \\
& t,
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[r, "t\eta^t"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "(t.r)"] & t^2 \arrow[d, "\mu^t"] \\
& t.
\end{tikzcd}$$

The 2-cells μ^t and η^t are called the **multiplication** and **unit** of t. We will often abbreviate the triple (t, μ^t, η^t) to just t.

Example 2.3.2. Of course, a monad in CAT is the familiar notion of a monad on a category. Varying the 2-category $\mathcal{K}$ we have:

- A monad in $\mathcal{V}$-CAT is a $\mathcal{V}$-enriched monad.
- A monad in the bicategory B$\mathcal{V}$ of Example 2.1.4 is a monoid in $\mathcal{V}$.
- A monad in the bicategory $\mathcal{R}$ of Example 2.1.5 on a commutative ring A is an associative unital A-algebra.
- A monad $\Phi\colon \mathcal{A} \nrightarrow \mathcal{A}$ in $\mathcal{V}$-Prof is called a promonad. The multiplication and unit give maps

  $$\Phi(b,c) \otimes \Phi(a,b) \longrightarrow \Phi(a,c) \qquad \text{and} \qquad \mathcal{A}(a,b) \longrightarrow \Phi(a,b)$$

  for $a, b, c \in \mathcal{A}$. This amounts to a $\mathcal{V}$-category $\mathcal{A}^\Phi$ with $\mathcal{A}^\Phi(a,b) = \Phi(a,b)$ and an identity-on-objects $\mathcal{V}$-functor $\mathcal{A} \to \mathcal{A}^\Phi$.

There are three different notions of morphisms between monads.

Definition 2.3.3. Let t and s be monads on x and y, respectively. A **lax monad functor** $(x, t) \to (y, s)$ is a pair (g, φ) where $g\colon x \to y$ is a 1-cell, and $\varphi\colon sg \to gt$ is a 2-cell such that the following commute:

$$\begin{tikzcd}
s^2g \arrow[r, "s\varphi"] \arrow[d, "\mu^s g"'] & sgt \arrow[r, "\varphi t"] \arrow[d, phantom, "(\varphi.\mu)"] & gt^2 \arrow[d, "g\mu^t"] \\
sg \arrow[rr, "\varphi"'] & & gt,
\end{tikzcd}
\qquad
\begin{tikzcd}
& g \arrow[dl, "\eta^s g"'] \arrow[dr, "g\eta^t"] \arrow[d, phantom, "(\varphi.\eta)"] & \\
sg \arrow[rr, "\varphi"'] & & gt.
\end{tikzcd}$$

A **colax monad functor** $(x,t) \to (y,s)$ is a lax monad functor $(y,s) \to (x,t)$ in $\mathcal{K}^{\mathrm{op}}$. Explicitly, it is a pair (h,ψ) where $h\colon x \to y$ is a 1-cell, and $\psi\colon ht \to sh$ is a 2-cell such that the following commute:

$$\begin{tikzcd}
ht^2 \arrow[r, "\psi t"] \arrow[d, "h\mu^t"'] & sht \arrow[r, "s\psi"] \arrow[d, phantom, "(\psi.\mu)"] & s^2h \arrow[d, "\mu^s h"] \\
ht \arrow[rr, "\psi"'] & & sh,
\end{tikzcd}
\qquad
\begin{tikzcd}
& h \arrow[dl, "h\eta^t"'] \arrow[dr, "\eta^s h"] \arrow[d, phantom, "(\psi.\eta)"] & \\
ht \arrow[rr, "\psi"'] & & sh.
\end{tikzcd}$$

If $x = y$, a **monad map** $t \to s$ is a 2-cell $\theta\colon t \to s$ such that $(1_x, \theta)$ is a colax monad functor. Explicitly, θ must make the following commute:

$$\begin{tikzcd}
t^2 \arrow[r, "\theta\theta"] \arrow[d, "\mu^t"'] \arrow[dr, phantom, "(\theta.\mu)"] & s^2 \arrow[d, "\mu^s"] \\
t \arrow[r, "\theta"'] & s,
\end{tikzcd}
\qquad
\begin{tikzcd}
& x \arrow[dl, "\eta^t"'] \arrow[dr, "\eta^s"] \arrow[d, phantom, "(\theta.\eta)"] & \\
t \arrow[rr, "\theta"'] & & s.
\end{tikzcd}$$

Let $\mathsf{Lax}(t,s)$, $\mathsf{Colax}(t,s)$ and $\mathsf{Mnd}_x(t,s)$ be the collections of lax monad functors, colax monad functors and monad maps from (x,t) to (y,s), with $x = y$ in the last case. Let $\mathsf{Lax}_g(t,s) \subseteq \mathsf{Lax}(t,s)$ consist of those lax monad functors whose 1-cell part is g, and similarly for $\mathsf{Colax}_g(t,s)$. It follows by definition that

$$\mathsf{Mnd}_x(t,s) = \mathsf{Colax}_x(t,s) = \mathsf{Lax}_x(s,t).$$

Example 2.3.4. If t is a monad on x, then $\eta^t\colon 1_x \to t$ is a monad map. The pair (t, μ^t) is both a lax monad functor $(x, 1_x) \to (x,t)$ and a colax monad functor $(x,t) \to (x, 1_x)$.

Example 2.3.5. If $f \dashv g\colon y \to x$ is an adjunction in $\mathcal{K}$, with unit η and counit ϵ, then gf becomes a monad on x with unit η and multiplication $g\epsilon f$. A straightforward verification gives that $(g, g\epsilon)\colon (y, 1_y) \to (x, gf)$ is a lax monad functor.

Given $(g,\varphi), (h,\psi) \in \mathsf{Lax}(t,s)$, a **lax monad functor transformation** $(g,\varphi) \to (h,\psi)$ is a 2-cell $\sigma\colon g \to h$ such that

$$\begin{tikzcd}
sg \arrow[r, "s\sigma"] \arrow[d, "\varphi"'] & sh \arrow[d, "\psi"] \\
gt \arrow[r, "\sigma t"'] & ht
\end{tikzcd} \tag{2.8}$$

commutes. Equipping $\mathsf{Lax}(t,s)$ with these transformations as morphisms makes it into a category.

Given lax monad functors $(g,\varphi)\colon (x,t) \to (y,s)$ and $(h,\psi)\colon (y,s) \to (z,r)$, the composite pair $(hg, h\varphi \cdot \psi g)$ is a lax monad functor $(x,t) \to (z,r)$. One can extend this to lax monad functor transformations to form a 2-category $\mathsf{MND}(\mathcal{K})$ whose objects are pairs (x, t) of a 0-cell x in $\mathcal{K}$ and a monad t on x, and where

$$\mathsf{MND}(\mathcal{K})((x,t),(y,s)) = \mathsf{Lax}(t,s).$$

The category of monads on a 0-cell x of $\mathcal{K}$ is Mnd_x, where objects are monads on x and morphisms are monad maps. It is precisely the subcategory of $\mathsf{MND}(\mathcal{K}^{\mathrm{op}})^{\mathrm{op}}$ on the monads on x and the 1-cells whose first component is 1_x. This is isomorphic to the category of monoids in the monoidal category $(\mathcal{K}(x,x), \circ, 1_x)$. As with all categories of monoids, we have:

Lemma 2.3.6. *The forgetful functor* $\mathsf{Mnd}_x \to \mathcal{K}(x,x)$ *creates limits.*

The most important object attached to a monad T on a category $\mathcal{C}$ is its Eilenberg–Moore category $\mathcal{C}^T$: the category of T-algebras. One approach to generalise this to monads in 2-categories is to follow the philosophy of generalised elements. If t is a monad on x, then a **t-algebra (of shape y)** is a pair (a, α) where $a \colon y \to x$ is a 1-cell and $\alpha \colon ta \to a$ is a 2-cell such that

$$\begin{tikzcd} t^2a \arrow[r, "t\alpha"] \arrow[d, "\mu^t a"'] & ta \arrow[d, "\alpha"] \\ ta \arrow[r, "\alpha"'] & a \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} a \arrow[r, "\eta^t a"] \arrow[dr, equal] & ta \arrow[d, "\alpha"] \\ & a \end{tikzcd}$$

commute. These are in fact the algebras for the monad $t_\circ$ on the category $\mathcal{K}(y,x)$, or equivalently they are the lax monad functors $(y,1) \to (x,t)$. In CAT, the traditional category of algebras is obtained by considering t-algebras whose shape is the terminal category.

Street gave a very elegant way to internalise this within $\mathcal{K}$, so that instead of a category of t-algebras of each possible shape, we can speak of an object of $\mathcal{K}$ acting as the object of t-algebras. There is an obvious 2-functor $\mathrm{Und} \colon \mathsf{MND}(\mathcal{K}) \to \mathcal{K}$ which forgets the monad structure. It has a right 2-adjoint $\mathrm{Id} \colon \mathcal{K} \to \mathsf{MND}(\mathcal{K})$, which is given by sending a 0-cell to the identity monad on it. We say that $\mathcal{K}$ **admits the construction of algebras** when Id has a further right 2-adjoint $\mathrm{Alg} \colon \mathsf{MND}(\mathcal{K}) \to \mathcal{K}$.

If t is a monad on x, we write $\mathrm{Alg}(x,t)$ as x^t and call it the **Eilenberg–Moore object** of t. We write the counit of the adjunction at (x,t) as $(u^t \colon x^t \to x, \chi^t \colon tu^t \to u^t)$. If $(g,\varphi) \colon (x,t) \to (y,s)$ is a lax monad functor, we write $\mathrm{Alg}(g,\varphi)$ as $g^\varphi \colon x^t \to y^s$. The adjunction ensures an isomorphism of categories

$$\mathcal{K}(y, x^t) \cong \mathsf{MND}(\mathcal{K})((y,1_y),(x,t)) = \mathsf{Lax}(y,t), \tag{2.9}$$

so that x^t represents the functor that sends y to the category of t-algebras of shape y.

The naturality of the counit of the adjunction gives, for each lax monad functor

(g, φ), commutative diagrams

$$\begin{tikzcd}
(x^t, 1) \arrow[r, "{(g^\varphi, 1)}"] \arrow[d, "{(u^t, \chi^t)}"'] & (y^s, 1) \arrow[d, "{(u^s, \chi^s)}"] \\
(x, t) \arrow[r, "{(g, \varphi)}"'] & (y, s)
\end{tikzcd}
\xmapsto{\mathrm{Und}}
\begin{tikzcd}
x^t \arrow[r, "g^\varphi"] \arrow[d, "u^t"'] & y^s \arrow[d, "u^s"] \\
x \arrow[r, "g"'] & y
\end{tikzcd}
\tag{2.10}$$

in $\mathsf{MND}(\mathcal{K})$ and $\mathcal{K}$, respectively. This being a 2-adjunction, there is a similar commutative diagram for 2-cells. In fact, Lack and Street show in [41, §2.2] that:

Theorem 2.3.7. *If $\mathcal{K}$ admits the construction of algebras, the assignment $(x, t) \mapsto u^t$ and $(g, \varphi) \mapsto (g^\varphi, g)$ extends to a locally fully faithful 2-functor*

$$\mathsf{MND}(\mathcal{K}) \hookrightarrow \mathcal{K}^{\to},$$

where $\mathcal{K}^{\to}$ is the arrow 2-category of $\mathcal{K}$.

Hence, every 1-cell g^φ of $\mathcal{K}$ fitting into the right-hand diagram of (2.10) corresponds to a unique lax monad functor (g, φ) on the left. The 1-cells equivalent to those of the form u^t in $\mathcal{K}^{\to}$ are called **monadic**. It follows that $\mathsf{MND}(\mathcal{K})$ is biequivalent to the full sub-2-category of $\mathcal{K}^{\to}$ on the monadic 1-cells. One should see this as evidence that the definition of $\mathsf{MND}(\mathcal{K})$ is meaningful.

Example 2.3.8. The 2-category CAT admits the construction of algebras. If T is a monad on $\mathcal{A}$, then $\mathrm{Alg}(\mathcal{A}, T) = \mathcal{A}^T$ is the Eilenberg–Moore category of T and the functor part of the counit at $(\mathcal{A}, T)$ is exactly $U^T \colon \mathcal{A}^T \to \mathcal{A}$, i.e. the classical forgetful functor from the category of T-algebras. By the previous theorem, lax monad functor $(G, \varphi) \colon (\mathcal{A}, T) \to (\mathcal{B}, S)$ are in bijection with functors $G^\varphi \colon \mathcal{A}^T \to \mathcal{B}^S$ such that $GU^T = U^S G^\varphi$.

Example 2.3.9. The 2-category $\mathsf{CAT}^{\mathrm{op}}$ also admits the construction of algebras. Note that monads in $\mathsf{CAT}^{\mathrm{op}}$ are the same as monads in CAT, because the categories of endo-1-cells are unchanged. In this case, $\mathrm{Alg}(\mathcal{C}, T) = \mathcal{C}_T$ is the Kleisli category of T and the functor part of the counit at $(\mathcal{C}, T)$ is the left adjoint $F_T \colon \mathcal{C} \to \mathcal{C}_T$ of the Kleisli adjunction for T. By the dual of the previous theorem, lax monad functors $(G, \psi) \colon (\mathcal{A}, T) \to (\mathcal{B}, S)$ are in bijection with functors $G_\psi \colon \mathcal{A}_T \to \mathcal{B}_S$ such that $F_S G = G_\psi F_T$.

In particular, the data of a map $\theta \colon T \to S$ of monads on $\mathcal{A}$ is both equivalent to the data of a functor $U^\theta \colon \mathcal{A}^S \to \mathcal{A}^T$ over $\mathcal{A}$ and to the data of a functor $F_\theta \colon \mathcal{A}_T \to \mathcal{A}_S$ under $\mathcal{A}$.

We saw in Example 2.3.5 that every adjunction in $\mathcal{K}$ induces a monad. In general, not every monad is induced by an adjunction, but this is the case if $\mathcal{K}$ admits the construction of algebras. Indeed, transporting the lax monad functor $(t, \mu^t) \colon (x, 1_x) \to (x, t)$ along the adjunction (2.9) gives a unique 1-cell $f^t \colon x \to x^t$ such that $u^t f^t = t$ and $\chi^t f^t = \mu^t$. One then shows that $f^t \dashv u^t$ and that this adjunction induces the monad t.

Moreover, if $f \dashv g\colon y \to x$ induces the monad t, then $(g, g\epsilon) \in \mathsf{Lax}(y,t)$ of Example 2.3.5 gives a unique comparison 1-cell $k\colon y \to x^t$ such that $u^t k = g$ and $u^t k\epsilon = u^t \epsilon k$. This is the **Eilenberg–Moore comparison 1-cell**. Dually, one obtains the Kleisli comparison 1-cell when $\mathcal{K}^{\mathrm{op}}$ admits the construction of algebras.

As was discussed in Section 1.2 in the case where $\mathcal{K} = \mathsf{CAT}$, it is not just right adjoints that induce monads in a 2-category. Given a 1-cell $g\colon y \to x$, if $\mathrm{ran}_g\, g$ exists then it has a canonical monad structure, called the **codensity monad** of g. When $\mathcal{K}$ admits the construction of algebras, the right adjoint $u^{\mathrm{ran}_g g}$ gives the **monadic reflection** of g in the sense of the following theorem. Let $(\mathcal{K}_0/x)^{\mathrm{ran}}$ be the category whose objects are 1-cells g in $\mathcal{K}$ with codomain x such that $\mathrm{ran}_g\, g$ exists, and whose morphisms are commutative triangles over x. Then:

Theorem 2.3.10. *If $\mathcal{K}$ admits the construction of algebras, the fully faithful functor*

$$\mathsf{Mnd}_x^{\mathrm{op}} = \mathsf{Lax}_x \xhookrightarrow{u^-} (\mathcal{K}_0/x)^{\mathrm{ran}}$$

given by restricting the 2-functor of Theorem 2.3.7 has a left adjoint, which sends $g\colon y \to x$ to its codensity monad.

Note that this is an adjunction of ordinary categories, not of 2-categories. In simple terms, it says that any commuting triangle on the left, factorises uniquely through a pair of commutative triangles on the right:

$$\begin{tikzcd}
y \arrow[rr, "h"] \arrow[dr, "g"'] & & x^t \arrow[dl, "u^t"] \\
& x, &
\end{tikzcd}
\qquad
\begin{tikzcd}
y \arrow[r, "k"] \arrow[dr, "g"'] & x^{\mathrm{ran}_g g} \arrow[r, dashed, "\widehat{h}"] \arrow[d, "u^{\mathrm{ran}_g g}"] & x^t \arrow[dl, "u^t"] \\
& x, &
\end{tikzcd}$$

where $k\colon y \to x^{\mathrm{ran}_g g}$ is given by the unit of the adjunction.

We will need to generalise one last notion from the classical theory of monads on categories to monads in a 2-category: the distributive laws of Beck [10].

Definition 2.3.11. Let s and t be monads on x. A **distributive law** of t over s is a 2-cell $\delta\colon ts \to st$ such that (s, δ) is a lax monad endofunctor of t and (t, δ) is a colax monad endofunctor of s. Explicitly, δ must make the following commute:

$$\begin{tikzcd}
ts^2 \arrow[r, "\delta s"] \arrow[d, "t\mu^s"'] & sts \arrow[r, "s\delta"] \arrow[d, phantom, "(\delta.\mu^s)"] & s^2t \arrow[d, "\mu^s t"] \\
ts \arrow[rr, "\delta"'] & & st,
\end{tikzcd}
\qquad
\begin{tikzcd}
& t \arrow[dl, "t\eta^s"'] \arrow[dr, "\eta^s t"] \arrow[d, phantom, "(\delta.\eta^s)"] & \\
ts \arrow[rr, "\delta"'] & & st,
\end{tikzcd}$$

$$\begin{tikzcd}
t^2s \arrow[r, "t\delta"] \arrow[d, "\mu^t s"'] & tst \arrow[r, "\delta t"] \arrow[d, phantom, "(\delta.\mu^t)"] & st^2 \arrow[d, "s\mu^t"] \\
ts \arrow[rr, "\delta"'] & & st,
\end{tikzcd}
\qquad
\begin{tikzcd}
& s \arrow[dl, "\eta^t s"'] \arrow[dr, "s\eta^t"] \arrow[d, phantom, "(\delta.\eta^t)"] & \\
ts \arrow[rr, "\delta"'] & & st.
\end{tikzcd}$$

Remark 2.3.12. Note that what we call a distributive law of t over s is sometimes called a distributive law of s over t in the literature, e.g. by Garner in [28, Def. 4]. We choose to follow Beck's original terminology in [10], bearing in mind that he writes composition of functors in diagrammatic order, i.e. opposite to our convention for 1-cells. This convention agrees with the maxim 'multiplication distributes *over* addition', which gives a distributive law of the free (multiplicative) monoid monad on Set *over* the free (additive) abelian group monad.

As is the case for Beck's notion, distributive laws in $\mathcal{K}$ allow us to compose monads.

Theorem 2.3.13. *Let t and s be monads on the 0-cell x of $\mathcal{K}$. Then there is a bijection between*

(i) *distributive laws $\delta\colon ts \to st$;*

(ii) *2-cells $\mu\colon (st)^2 \to st$ such that*

  (a) *$(st, \mu, \eta^s\eta^t)$ is a monad,*
  (b) *the 2-cells $s\eta^t\colon s \to st$ and $\eta^s t : t \to st$ are monad maps,*
  (c) *the* **middle unitary law** *holds, i.e. the following diagram commutes:*

$$\begin{tikzcd}
 & st \arrow[dl, "s\eta^t\eta^s t"'] \arrow[dr, equal] & \\
stst \arrow[rr, "\mu"'] & & st.
\end{tikzcd}$$

The ***composite monad*** *structure on st obtained from a distributive law δ in this way is denoted by $s \circ_\delta t$.*

Proof. The equivalence between (i) and (ii) is proved for $\mathcal{K} = \mathsf{CAT}$ in Beck's work introducing distributive laws [10, §1]. The proof in a general 2-category is identical. The multiplication of $s \circ_\delta t$ is the composite 2-cell

$$stst \xrightarrow{s\delta t} s^2t^2 \xrightarrow{\mu^s\mu^t} st.$$

Conversely, one obtains a distributive law as the composite

$$ts \xrightarrow{\eta^s ts\eta^t} stst \xrightarrow{\mu} st. \qquad \square$$

Note that in (ii) we need not have specified the unit of the composite monad as $\eta^s\eta^t$, since this is necessary from the fact that $s\eta^t$ and $\eta^s t$ are monad maps.

Examining the first two diagrams in Definition 2.3.11 together with (2.8), one sees that a lax monad endofunctor (s, δ) of t gives a distributive law iff η^s and μ^s are lax monad functor transformations

$$1_{(x,t)} \longrightarrow (s,\delta) \qquad \text{and} \qquad (s,\delta)\circ(s,\delta) = (s^2, s\delta\cdot\delta s) \longrightarrow (s,\delta),$$

respectively. In other words, δ is a distributive law iff $((s,\delta),\mu^s,\eta^s)$ is a monad in $\mathsf{MND}(\mathcal{K})$ on the 0-cell (x,t).

By Theorem 2.3.7, if $\mathcal{K}$ admits the construction of algebras, such a monad amounts to a monad on $u^t\colon x^t \to x$ in $\mathcal{K}^{\to}$ whose underlying monad on x is s. Explicitly, this is a monad $\overline{s}$ on x^t such that

$$u^t\overline{s} = su^t, \qquad u^t\eta^{\overline{s}} = \eta^s u^t \qquad \text{and} \qquad u^t\mu^{\overline{s}} = \mu^s u^t.$$

We call this a **monad lift** of s along u^t. Note that these conditions are equivalent to requiring that $(u^t, 1)\colon (x^t, \overline{s}) \to (x, s)$ be a lax monad functor.

Theorem 2.3.14. *If $\mathcal{K}$ admits the construction of algebras, then distributive laws $\delta\colon ts \to st$ are in bijection with monad lifts of s along u^t.*

Dually, if $\mathcal{K}^{\mathrm{op}}$ admits the construction of algebras, a distributive law $\delta\colon ts \to st$ amounts to a monad extension of t along $f_s\colon x \to x_s$, where x_s is the Kleisli object of s.

The assignment of Theorem 2.3.13 of a distributive law to its composite monad extends to a 2-functor Cmp: $\mathsf{MND}^2(\mathcal{K}) \to \mathsf{MND}(\mathcal{K})$, which is in fact the component at $\mathcal{K}$ of the multiplication of a 2-monad MND on the 2-category of 2-categories. The component at $\mathcal{K}$ of the unit of this 2-monad is $\mathrm{Id}\colon \mathcal{K} \to \mathsf{MND}(\mathcal{K})$.

If $\delta\colon ts \to st$ is a distributive law and $\mathcal{K}$ admits the construction of algebras, then there is an isomorphism between the Eilenberg–Moore objects of $s \circ_\delta t$ and of the monad lift $\overline{s}$ of s along u^t that δ induces. This follows from the Yoneda lemma, because $x^{s\circ_\delta t}$ and $(x^t)^{\overline{s}}$ represent the same functor. Indeed, the categories of $(s \circ_\delta t)$- and $\overline{s}$-algebras of shape y are isomorphic by Beck's characterisation in the classical case, since $(s \circ_\delta t)_\circ = s_\circ \circ_{\delta_\circ} t_\circ$.

In fact, Beck gave a third characterisation of these algebras. An $(s \circ_\delta t)$-algebra of shape y amounts to a triple (a, α^s, α^t) where $a\colon y \to x$ is a 1-cell and $\alpha^s\colon sa \to a$ and $\alpha^t\colon ta \to a$ are 2-cells such that:

- (a, α^s) is an $s_\circ$-algebra,
- (a, α^t) is a $t_\circ$-algebra,
- the following diagram commutes:

$$\begin{tikzcd}
tsa \arrow[rr, "\delta a"] \arrow[d, "t\alpha^s"'] & & sta \arrow[d, "s\alpha^t"] \\
ta \arrow[r, "\alpha^t"'] & a & sa. \arrow[l, "\alpha^s"]
\end{tikzcd}$$

Moreover, the category of $(s\circ_\delta t)$-algebras of shape y is isomorphic to the category of $\overline{s}$-algebras of the same shape.

In particular, note that if $\mathcal{K}$ admits the construction of algebras then $(\mathcal{K}, \mathrm{Alg})$ is a pseudoalgebra for the 2-monad MND, since we have just seen that

$$\begin{tikzcd}
\mathsf{MND}^2(\mathcal{K}) \arrow[r, "\mathsf{MND}(\mathrm{Alg})"] \arrow[d, "\mathrm{Comp}"'] & \mathsf{MND}(\mathcal{K}) \arrow[d, "\mathrm{Alg}"] \\
\mathsf{MND}(\mathcal{K}) \arrow[r, "\mathrm{Alg}"'] & \mathcal{K}
\end{tikzcd}$$

commutes up to isomorphism, as does the triangle in the unit axiom.

In [41], Lack and Street defined an alternative 2-category of monads in $\mathcal{K}$, called $\mathsf{EM}(\mathcal{K})$. It arose from the realisation that the Eilenberg–Moore object x^t, being defined in terms of the category of 1-cells into it, is in fact a weighted limit. In particular, let Δ_a be the augmented simplex category, consisting of finite ordinals and monotone maps between them. Under ordinal sum, it is the free monoidal category containing a monoid. It follows that a 2-functor $\mathrm{B}\Delta_a \to \mathcal{K}$ is precisely a monad t in $\mathcal{K}$. The algebra object x^t for this monad is then the limit of this 2-functor weighted by a fixed 2-functor $W \colon \mathrm{B}\Delta_a \to \mathsf{Cat}$. The 2-category $\mathsf{EM}(\mathcal{K})$ is defined as the free completion of $\mathcal{K}$ under W-weighted limits. It has the same 0- and 1-cells as $\mathsf{MND}(\mathcal{K})$, but more 2-cells in general.

2.4 Monads in CAT and on Set

Monads in CAT are of course the prototype for the general theory of monads. As we saw in Example 2.3.8, both CAT and $\mathsf{CAT}^{\mathrm{op}}$ admit the construction of algebras, recovering the classical Eilenberg–Moore and Kleisli categories for monads. In this context, many natural questions one can ask about monadic functors and categories of algebras have well-known answers, some of which we record here for later use. We also focus on the case of monads on Set, giving several examples.

We begin by recording Beck's monadicity theorems. The first can be found in his original paper [11, Thm. 4.1]. The second is stated as in Barr and Wells's book [9, §3.5]. Recall that a functor $U \colon \mathcal{B} \to \mathcal{A}$ **creates** the colimit of a diagram $D \colon \mathcal{I} \to \mathcal{B}$ if, given a colimit cocone $\lambda \colon UD \to L$ in $\mathcal{A}$, there is a cocone $\mu \colon D \to M$ in $\mathcal{B}$ such that $U\mu$ is isomorphic to λ, and any such cocone is a colimit cocone. In particular, it reflects colimits for D, and preserves them as soon as UD has a colimit.

Theorem 2.4.1 (Monadicity)**.** *A functor $U \colon \mathcal{B} \to \mathcal{A}$ is monadic iff it is a right adjoint and it creates U-split coequalisers.*

Theorem 2.4.2 (Crude Monadicity)**.** *Let $\mathcal{B}$ be a category with reflexive coequalisers and $U \colon \mathcal{B} \to \mathcal{A}$ be a right adjoint which preserves them and which reflects isomorphisms. Then U is monadic.*

If $\theta \colon T \to S$ is a map of monads on a category $\mathcal{A}$, we write $U^\theta \colon \mathcal{A}^S \to \mathcal{A}^T$ for the functor between categories of algebras it induces. By Theorem 2.3.7, this makes $U^- \colon \mathsf{Mnd}_{\mathcal{A}}^{\mathrm{op}} \to \mathsf{CAT}/\mathcal{A}$ into a fully faithful functor whose essential image consists of the monadic functors. We call a functor of the form U^θ **algebraic**.

Proposition 2.4.3. *Let $\theta \colon T \to S$ be a morphism of monads on $\mathcal{A}$.*

(i) *The algebraic functor $U^\theta \colon \mathcal{A}^S \to \mathcal{A}^T$ creates U^S-split coequalisers.*

(ii) *U^θ is monadic iff it has a left adjoint.*

Proof. Since $U^T U^\theta = U^S$, if a parallel pair (f, g) in $\mathcal{A}^S$ is U^S-split, then $(U^\theta f, U^\theta g)$ is U^T-split. The first statement then follows from the fact that U^S and U^T create U^S- and U^T-split coequalisers, respectively.

The second statement is then obvious from the Monadicity Theorem. □

The following theorem of Linton [46, Cor. 1] gives sufficient conditions for algebraic functors to have left adjoints, and hence be monadic.

Theorem 2.4.4. *Let T be a monad on $\mathcal{A}$ such that $\mathcal{A}^S$ has reflexive coequalisers.*

(a) *For every monad map $\theta\colon T \to S$, the algebraic functor $U^\theta\colon \mathcal{A}^S \to \mathcal{A}^T$ has a left adjoint, denoted by F^θ, whose value at the object $(A, \alpha) \in \mathcal{A}^T$ is given by the coequaliser in $\mathcal{A}^S$ of the reflexive pair*

$$\begin{tikzcd} F^S TA \arrow[r, "F^S\theta"] \arrow[rr, bend right, "F^S\alpha"'] & F^S SA \arrow[r, "\mu^S_A"] & F^S A. \end{tikzcd}$$

(b) *If $\mathcal{A}$ has coproducts, then $\mathcal{A}^T$ is cocomplete.*

We now focus on monads on Set. The next theorem follows from Borceux's [15, Vol. II, Thm. 4.3.5], and is a good example of how arbitrary monads on Set are especially well behaved.

Theorem 2.4.5. *Let T be a monad on* Set. *Then* Set^T *is complete and cocomplete. In particular, all algebraic functors over* Set *are monadic.*

Proposition 2.4.6. *Let $U\colon \mathcal{A} \to \mathsf{Set}$ be a functor and $U^X\colon \mathcal{A} \to \mathsf{Set}$ be given by $U^X(A) = (UA)^X$ for a set X. If U is monadic, then so is U^X.*

Proof. We may assume that $U = U^T$ for some monad T on Set. By the proof of Borceux's [15, Vol. II, Thm. 4.4.5], it suffices to show that $F^T X$ is a projective regular generator of Set^T. It is projective because a coproduct of projective objects is projective, $F^X \cong \sum_X F^T 1$ and $F^T 1$ is projective. It is a strong generator because $\mathsf{Set}^T(F^T X, -) \cong \prod_X U^T$ reflects isomorphisms, by Corollary 4.5.11 in Borceux's book. This is enough because Set^T is a regular category. ☐

Recall that algebraic theories whose abstract operations have arity less than some regular cardinal κ correspond to monads on Set preserving κ-filtered colimits.

Definition 2.4.7. The **rank** of a monad T on Set is the smallest regular cardinal κ such that T preserves κ-filtered colimits. If no such cardinal exists, we say that T has **no rank**.

Finitary monads are hence those with rank $\aleph_0$. Monads with rank are particularly well behaved, e.g. their categories of algebras are locally presentable. However, not every monad on Set has rank, as we will see in the examples below.

There are two monads on Set which are exceptional in many ways. We write $g \cdot f$ or simply gf for the composite of two functions $f\colon X \to Y$ and $g\colon Y \to Z$.

Lemma 2.4.8. *The following are equivalent for a monad T on* Set*:*

(i) *there is a T-algebra with at least 2 elements;*

(ii) *the unit η^T is (pointwise) monic.*

Proof. If $\eta^T_2 \colon 2 \to T2$ is monic then there is clearly a T-algebra with at least 2 elements. Conversely, let (A, α) be a T-algebra with at least 2 elements and $B \in \mathsf{Set}$. For any $b_1 \neq b_2 \in B$ there is a function $f \colon B \to A$ such that $f(b_1) \neq f(b_2)$. Since $f = \alpha \cdot Tf \cdot \eta^T_B$, it follows that $\eta^T_B(b_1) \neq \eta^T_B(b_2)$. □

A monad on Set which satisfies these conditions is called **consistent**. If T is an inconsistent monad, then every T-algebra is either empty or a singleton. Unsurprisingly, there are only two such monads up to isomorphism.

Example 2.4.9. The terminal object of $[\mathsf{Set}, \mathsf{Set}]$ is the functor constant at 1. As the terminal object in a monoidal category, it has a unique monoid structure. This gives the terminal monad T on Set.

It is of course finitary: it corresponds to the theory presented by a single constant and the equation $x = y$. Its category of algebras is hence equivalent to the terminal category, with the forgetful functor $1 \to \mathsf{Set}$ picking out 1. It is one of the only three idempotent monads on Set.

Example 2.4.10. The monad T has a unique submonad T_0 with $\mathsf{T}_0\varnothing = \varnothing$. It is again finitary, corresponding to the theory with no operations and the equation $x = y$. Its category of algebras is equivalent to the walking arrow category, which embeds as $\{\varnothing < 1\} \subseteq \mathsf{Set}$. It is the remaining idempotent monad on Set, once we account for the identity.

Proposition 2.4.11. *If T is a consistent monad on* Set*, then both F^T and T reflect isomorphisms.*

Proof. Since $T = U^T F^T$ and U^T reflects isomorphisms, it suffices to show that T does too. Let $f \colon X \to Y$ be a function such that Tf is invertible. Since $\eta^T_Y \cdot f = Tf \cdot \eta^T_X$ is injective because T is consistent, f is injective.

For $A \subseteq X$, let $[A] \colon X \to 2 = \{0, 1\}$ denote the indicator function of A, and similarly for $B \subseteq Y$. Note that $[\operatorname{im} f]f = [X]$. Applying T,

$$T[\operatorname{im} f] \cdot Tf = T[X],$$

and since Tf is invertible this shows that $T[\operatorname{im} f]$ factors through $T[X]$, which itself factors through T applied to the inclusion $t \colon \{1\} \to 2$. Now $T[\operatorname{im} f]$ sends $\eta^T_Y(y)$ to $\eta^T_2(1)$ if $y \in \operatorname{im} f$ and to $\eta^T_2(0)$ otherwise. Hence, to show that f is surjective, it suffices to show that $\eta^T_2(0)$ is not in the image of Tt.

A T-algebra map $g \colon F^T 2 \to A$ is determined by $g \cdot \eta^T_2 \colon 2 \to A$. If $\eta^T_2(0)$ is in the image of Tt, then this is further determined by $g \cdot Tt$, which is itself determined by where $\eta^T_1(1)$ is sent to. This gives a contradiction by taking A with at least two elements and varying the value of $g \cdot \eta^T_2$ at 0 but not at 1. □

We now introduce a number of consistent monads on Set which we will use as examples in later chapters. The reason for this particular collection of examples is that each of these monads restricts to a monad on finite sets. We write X^Y for the set of functions from a set Y to a set X.

Example 2.4.12. Any set E is a monoid in $(\mathsf{Set}, +, \varnothing)$ in a unique way, with unit $! \colon \varnothing \to 1$ and multiplication $\nabla_E \colon E + E \to E$. The functor $\mathsf{Set} \to [\mathsf{Set}, \mathsf{Set}]$

which sends X to $- + X$ sends binary coproducts to functor composition, and hence is strong monoidal. This makes $- + E$ a monad which we denote by E_E. In programming language semantics, this is called the **exception monad** because a function $X \to Y + E$ can be thought of as a function $X \to Y$ which may raise an exception $e \in E$.

This monad is always finitary. It corresponds to the theory with E constants and no equations. Its category of algebras is hence the category of E-pointed sets, which is equivalent to the slice category E/Set.

Example 2.4.13. Dually, any set I is a comonoid in $(\mathsf{Set}, \times, 1)$ in a unique way, with unit $!\colon I \to 1$ and comultiplication $\Delta_I\colon I \to I^2$. The Yoneda embedding $\mathsf{Set}^{\mathrm{op}} \to [\mathsf{Set}, \mathsf{Set}]$ sends binary products to functor composition since $\mathsf{Set}(X \times Y, -) \cong \mathsf{Set}(X, -) \circ \mathsf{Set}(Y, -)$, and hence is strong monoidal. This makes $(-)^I$ a monad which we denote by R_I. In programming language semantics, this is the **reader monad** because a function $X \to Y^I$ is a function $X \to Y$ which depends on a parameter $i \in I$ which it 'reads'. Note, in particular, that $\mathsf{R}_\varnothing = \mathsf{T}$ and $\mathsf{R}_1 = 1$.

This monad is finitary iff I is finitely presentable, which in Set is equivalent to being finite. In any case, it corresponds to the theory presented by a single hyperaffine I-ary operation, i.e. an I-ary operation θ such that

$$\theta(i \mapsto x) = x \qquad \text{and} \qquad \theta(i \mapsto \theta(j \mapsto x_{ij})) = \theta(i \mapsto x_{ii}).$$

We now take I to be a finite set n. Being a finite power of the identity monad, which is finitary and commutative, the category of algebras of R_n is described by Faro and Kelly in [26, Prop. 11]. It is the result of freely adjoining an initial object to $(\mathsf{Set}_{\geq 1})^n$, the n-th power of the category of nonempty sets. The forgetful functor sends the initial object to $\varnothing$, and $(X_i)_{i \in n}$ to $\prod_{i \in n} X_i$.

Example 2.4.14. The cartesian closed structure of Set gives an adjunction $- \times S \dashv (-)^S$ for any set S. This induces the monad $(S \times -)^S$ which we denote by S_S. In programming language semantics, this is the **state monad** because a function $X \to (S \times Y)^S$ can be thought of as a function $X \to Y$ which has access to a set of states S which it can 'read' and 'update'.

It is finitary iff S is finite, which we now assume. Its theory admits a presentation with a unary π_i operation for each $i \in S$ and an S-ary operation δ, subject the equations[1]

$$\begin{aligned} \pi_i \pi_j(x) &= \pi_j(x), \\ \delta(i \mapsto \pi_i(x)) &= x, \\ \pi_i(\delta(i \mapsto x_i)) &= \pi_i(x_i), \end{aligned}$$

for all $i, j \in S$. Clearly $\mathsf{S}_\varnothing = \mathsf{T}$. When $S \neq \varnothing$, its category of algebras is in fact equivalent to Set. Indeed, the right adjoint $(-)^S$ is monadic by Proposition 2.4.6.

Example 2.4.15. The cartesian closed structure of Set also implies that $S^-\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is adjoint to itself on the right. This induces the monad S^{S^-}

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[1] This presentation was shown to the author by Tom Leinster in a private communication.

which we denote by D_S and call the **endomorphism monad** of S. We have $\mathsf{D}_\varnothing = \mathsf{T}_0$ and $\mathsf{D}_1 = \mathsf{T}$. For $|S| > 1$, this monad is never finitary. In fact, it has no rank, as we will see in Theorem 6.1.24.

Taking $S = 2$ gives the familiar **double powerset monad**. The category of algebras of D_2 is equivalent to $\mathsf{Set}^{\mathrm{op}}$, which is itself equivalent to the category of complete atomic Boolean algebras by Stone duality. This is because the functor 2^-, i.e. the contravariant powerset functor, is monadic thanks to a famous theorem of Paré – see e.g. the book by Mac Lane and Moerdijk [49, Thm. IV.5.3]. The finitary part $\mathsf{D}_2^{\mathrm{f}}$ corresponds to the theory of all Boolean algebras.

Something similar happens for $2 < |S| < \infty$. The finitary part $\mathsf{D}_S^{\mathrm{f}}$ corresponds to the theory of S-valued Post algebras. One can adapt the proof of Paré's theorem to show that S^- is monadic in this case too, so that the D_S-algebras are those of S-valued functions on a set. Epstein [25, Thm. 18] showed that these are precisely the complete atomistic S-valued Post algebras.

Example 2.4.16. Let M be a monoid. Much like in Example 2.4.13, the functor $\mathsf{Set} \to [\mathsf{Set}, \mathsf{Set}]$ which sends X to $X \times -$ is strong monoidal. This makes $M \times -$ a monad which we denote by M_M and call the **free M-set monad**.

It is always finitary. Its theory is presented by a unary operation for each element of M, which compose exactly as the elements multiply in M. Its category of algebras is the category of M-sets, i.e. sets equipped with a left action by M. It is isomorphic to the functor category $[\mathrm{B}M, \mathsf{Set}]$, where $\mathrm{B}M$ denotes the one-object category corresponding to M.

Example 2.4.17. For a set X, let $\mathsf{P}X$ denote its powerset. Given a function $f \colon X \to Y$, let $\mathsf{P}f$ send a subset of X to its image in Y. This makes P into a functor, which in fact has a monad structure given by

$$\eta^{\mathsf{P}}_X(x) = \{x\} \qquad \text{and} \qquad \mu^{\mathsf{P}}_X(\mathfrak{A}) = \bigcup_{A \in \mathfrak{A}} A,$$

commonly called the **powerset monad**. Like D_S for $|S| > 1$, P does not have rank. Its finitary part is the **finite powerset monad** P^{f}, which sends X to the set of finite subsets of X.

The theory corresponding to P^{f} can be presented by a unital, associative, commutative and idempotent binary operation. Its category of algebras is hence the category of commutative idempotent monoids, which is isomorphic to the category of bounded join-semilattices. The category of algebras of P is the category of suplattices, whose objects are posets with arbitrary suprema (and hence also arbitrary infima) and whose morphisms are supremum-preserving monotone maps.

There is also the **nonempty powerset monad** $\mathsf{P}_+ \subseteq \mathsf{P}$ and the **nonempty finite powerset monad** $\mathsf{P}^{\mathrm{f}}_+ \subseteq \mathsf{P}^{\mathrm{f}}$.

For our last two examples of monads on Set we will need some basic theory of filters and ultrafilters on sets. This should come as no surprise, since the codensity monad of $J \colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$ is the ultrafilter monad (defined below), as shown by Kennison and Gildenhuys [37, p. 341]. The following definitions and properties are standard – see e.g. the book by Hofmann, Seal and Tholen [30, §II.1.12 and 13].

Definition 2.4.18. Let X be a set. A **filter** on X is a subset $\mathcal{F}$ of $\mathsf{P}X$ such that

(i) $X \in \mathcal{F}$;

(ii) for $A, B \subseteq X$, we have $A \cap B \in \mathcal{F}$ iff $A, B \in \mathcal{F}$[2].

A filter $\mathcal{F}$ is **proper** if it is a proper subset of $\mathsf{P}X$, equivalently if $\varnothing \notin \mathcal{F}$. An **ultrafilter** is a filter that also satisfies:

(iii) for $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$, but not both.

Ordering the filters on X by inclusion, ultrafilters are precisely the maximal proper ones.

Given $A \subseteq X$, the **principal filter** at A is the filter

$$\uparrow\! A = \{B \subseteq X \mid A \subseteq B\}.$$

This is an ultrafilter iff $A = \{x\}$ for some $x \in X$, in which case it is called the **principal ultrafilter** at x. Any filter on a finite set is principal; indeed, a filter is closed under finite intersections, and taking the intersection over the entire filter gives a least element.

An **ideal** on X is a subset $\mathcal{I}$ of $\mathsf{P}X$ such that

(iv) $\varnothing \in \mathcal{I}$;

(v) for $A, B \subseteq X$, we have $A \cup B \in \mathcal{I} \iff A, B \in \mathcal{I}$.

Ideals are the dual notion of filters. Given $A \subseteq X$, the **principal ideal** at A is the ideal

$$\downarrow\! A = \{B \subseteq X \mid B \subseteq A\}.$$

Note that the complement of an ultrafilter $\mathcal{U}$ is an ideal.

The most important result concerning ultrafilters is the ultrafilter lemma. It can be derived using the axioms of choice. It is the main way of constructing nonprincipal ultrafilters. For a proof, see [30, Cor. II.1.13.5].

Lemma 2.4.19 (Ultrafilter lemma)**.** *Let X be a set, $\mathcal{F}$ be a filter on X and $\mathcal{I}$ be an ideal on X such that $\mathcal{F} \cap \mathcal{I} = \varnothing$. Then there exists an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F} \subseteq \mathcal{U}$ and $\mathcal{U} \cap \mathcal{I} = \varnothing$.*

Example 2.4.20. Let $\mathsf{F}X$ denote the set of filters on a set X. Then F becomes a functor as follows: given a function $f\colon X \to Y$ and $\mathcal{F} \in \mathsf{F}X$, let

$$\mathsf{F}f(\mathcal{F}) = \{B \subseteq Y \mid f^{-1}B \in \mathcal{F}\}. \tag{2.11}$$

---

[2] This condition is often split into two: if $A \subseteq B \subseteq X$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$; and if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. The equivalence between the two definitions is straightforward.

It has a well-known monad structure, which we call the **filter monad**. The unit sends $x \in X$ to $\uparrow\{x\}$. Given $\mathfrak{F} \in \mathsf{F}^2 X$, we have

$$\mu^{\mathsf{F}}_X(\mathfrak{F}) = \{A \subseteq X \mid A^\sharp \in \mathfrak{F}\} \quad \text{where} \quad A^\sharp = \{\mathcal{F} \in \mathsf{F}X \mid A \in \mathcal{F}\}. \tag{2.12}$$

Much like P, the monad F does not have rank. Its finitary part is P^{f}. The category of F-algebras was identified by Day [16, Thms. 3.3 and 4.5] as the category of continuous lattices and maps preserving directed joins and arbitrary meets.

Let F_+X be the set of proper filters on a set X. It is easy to see that F_+ is a subfunctor of F. In fact, it is also a submonad, called the **proper filter monad**. The unit of F already factors through F_+. The multiplication becomes

$$\mu^{\mathsf{F}_+}_X(\mathfrak{F}) = \{A \subseteq X \mid A^\sharp \cap \mathsf{F}_+X \in \mathfrak{F}\}$$

for $\mathfrak{F} \in (\mathsf{F}_+)^2 X$. The resulting filter is proper, because $\varnothing^\sharp = \{\mathsf{P}X\}$ and hence $\varnothing^\sharp \cap \mathsf{F}_+X = \varnothing \notin \mathfrak{F}$.

Example 2.4.21. Let $\mathsf{U}X$ denote the set of ultrafilters on a set X. Then U becomes a submonad of F, which we call the **ultrafilter monad**. Its action on morphisms and unit are the same as those of F, noting that they produce ultrafilters. Given $\mathfrak{U} \in \mathsf{U}^2 X$, then

$$\mu^{\mathsf{U}}_X(\mathfrak{U}) = \{A \subseteq X \mid A^\sharp \cap \mathsf{U}X \in \mathfrak{U}\}. \tag{2.13}$$

Once again, U has no rank. Its category of algebras was famously identified by Manes [50, Prop. 5.5] as the category CHaus of compact Hausdorff spaces, with its natural underlying set functor. This is in stark contrast to the underlying set functor $\mathsf{Top} \to \mathsf{Set}$ which is as far from being monadic as possible, since it induces the identity monad. Thus the theory of compact Hausdorff spaces, unlike that of all topological spaces, is an algebraic one, albeit far from being finitary.

Briefly, every point x of a topological space X has a neighbourhood filter $\mathcal{N}_x$ consisting of all those subsets of X whose interior contains x. We say that an ultrafilter $\mathcal{U}$ on X **converges** to x if $\mathcal{N}_x \subseteq \mathcal{U}$. The space X is Hausdorff iff every ultrafilter converges to at most one point, and it is compact iff every ultrafilter converges to at least one point. This shows how to equip a compact Hausdorff space X with a U-algebra structure: to each $\mathcal{U} \in \mathsf{U}X$ one assigns the unique point of X to which $\mathcal{U}$ converges.

Chapter 3

Cauchy density

In his influential 1973 article [43], Lawvere showed how metric spaces can be seen as examples of categories enriched in the poset of extended reals $\mathbb{R}^+ = ([0, \infty], \geq)$ with its additive monoidal structure. This important example led to the definition of the Cauchy completion of an enriched category, as a generalisation of the completion of a metric space.

In page 155 of the same article, he gave a categorical characterisation of when a short map of metric spaces has a topologically dense image. He defined a $\mathcal{V}$-functor $F \colon \mathcal{A} \to \mathcal{B}$ to be $\mathcal{V}$-dense when the counit of the adjunction $F_* \dashv F^*$ in $\mathcal{V}\text{-}\mathsf{Prof}$ is invertible. The term dense has since taken a different (although related) meaning in category theory[1], and so we instead use the term **Cauchy dense** to refer to such $\mathcal{V}$-functors, following Day [17, p. 428].

Throughout this chapter and unless otherwise specified, every category, functor and natural transformation will be assumed to be $\mathcal{V}$-enriched, where $\mathcal{V}$ is a Bénabou cosmos, i.e. a complete and cocomplete symmetric monoidal closed category. All Kan extensions will be assumed to be pointwise. We write $(-)_0 \colon \mathcal{V}\text{-}\mathsf{CAT} \to \mathsf{CAT}$ for the underlying ordinary category 2-functor.

Lucatelli Nunes and Sousa [47] showed, using different terminology, that Cauchy dense functors between small categories coincide with the cofully faithful 1-cells of $\mathcal{V}\text{-}\mathsf{Cat}$, as well as with the absolutely (co)dense functors. In particular, every Cauchy dense functor is also dense. In Section 3.1, we review their characterisation and streamline some of the proofs using the proarrow equipment point of view. We will also briefly consider their behaviour under base change and the corresponding notion for functors between large categories.

Section 3.2 focuses on those Cauchy dense functors which are also fully faithful. These are exactly those 1-cells sent to equivalences by the pseudofunctor $(-)_* \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$. In the context of metric spaces, they correspond to embeddings of dense subspaces. Theorem 3.2.5 is the categorical generalisation of the fact that the completion of a metric space is the largest space into which it embeds as a dense subset. Such embeddings are replaced by fully faithful Cauchy dense functors and the completion of a metric space is replaced by the Cauchy completion of a category.

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[1] Incidentally, the significance of categorical density in the setting of metric spaces does not seem to have been considered in the literature.

We also give a characterisation in Theorem 3.2.7 of fully faithful Cauchy dense functors $F\colon \mathcal{A} \to \mathcal{B}$ as those for which $[F, \mathcal{C}]\colon [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$ is an equivalence for every Cauchy complete $\mathcal{C}$. A closely related result appears in work of Lucatelli Nunes, Prezado and Sousa [53, Thm. 2.5], of which the author was unaware at the time of writing.

Section 3.3 studies Cauchy dense functors in a number of special cases. Cauchy dense functors between one-object categories, i.e. monoids in $\mathcal{V}$, are shown to coincide with the epimorphisms of monoids when $\mathcal{V} = \mathsf{Set}$ and $\mathcal{V} = \mathsf{Ab}$. This is not the case in general, and a counterexample is provided where $\mathcal{V}$ the category of finitary endofunctors of Set. With $\mathcal{V} = \mathsf{Set}$, we show that Cauchy dense functors induce bijections between connected components, which allows us to characterise those Cauchy dense functors whose domain is a groupoid. Lastly, Theorem 3.3.20 shows that the condition in the definition of Cauchy density can be replaced by an apparently weaker one when working with split-full functors (Definition 3.2.2).

3.1 Cauchy dense functors

We begin by defining our object of study:

Definition 3.1.1 (Lawvere)**.** A functor $F\colon \mathcal{A} \to \mathcal{B}$ between small categories is **Cauchy dense** if the counit of the profunctor adjunction $F_* \dashv F^*$ is an isomorphism, i.e. if for every $b, b' \in \mathcal{B}$ the map

$$\epsilon_{b,b'}\colon \int^a \mathcal{B}(Fa, b') \otimes \mathcal{B}(b, Fa) \longrightarrow \mathcal{B}(b, b')$$

induced by the composition of $\mathcal{B}$ is an isomorphism in $\mathcal{V}$.

Recall from Lemma 2.2.10 that asking that the unit of $F_* \dashv F^*$ be an isomorphism is precisely asking that F be fully faithful. Being Cauchy dense is hence a dual condition.

The term ‘Cauchy dense’ is taken from Day [17, p. 428], and motivated by the following proposition.

Proposition 3.1.2. *An $\mathbb{R}^+$-functor $f\colon A \to B$ between classical metric spaces is Cauchy dense iff its image is topologically dense in B.*

Proof. The Cauchy density of f reduces to the condition

$$\inf_a (B(b, fa) + B(fa, b')) = B(b, b') \qquad \forall b, b' \in B.$$

Taking $b' = b$, we see that for any $\epsilon > 0$ there is some $a \in A$ with $B(b, fa) < \epsilon$. Hence, fA is dense in B.

Conversely, let $b, b' \in B$ and, for any $\epsilon > 0$, take $a \in A$ with $B(b, fa) < \epsilon$. Then

$$B(b, fa) + B(fa, b') \leq B(b, fa) + B(fa, b) + B(b, b') < 2\epsilon + B(b, b').$$

Since ϵ can be made arbitrarily small, we conclude that f is Cauchy dense. □

For more general $\mathcal{V}$, the Cauchy density of $F\colon \mathcal{A} \to \mathcal{B}$ does not only depend on its image, although it does if $\mathcal{V}_0$ is a preorder (Proposition 3.3.1 below).

Just as full faithfulness is a self-dual condition, so is Cauchy density.

Proposition 3.1.3. *A functor $F\colon \mathcal{A} \to \mathcal{B}$ is Cauchy dense iff $F^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is.*

Proof. For all $a \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, consider the commutative diagram

$$\begin{tikzcd}
\mathcal{B}(Fa, b') \otimes \mathcal{B}(b, Fa) \arrow[r, "M_{\mathcal{B}}"] \arrow[d, "\sigma"', "\cong"] & \mathcal{B}(b, b') \arrow[d, equal] \\
\mathcal{B}^{\mathrm{op}}(Fa, b) \otimes \mathcal{B}^{\mathrm{op}}(b', Fa) \arrow[r, "M_{\mathcal{B}^{\mathrm{op}}}"'] & \mathcal{B}^{\mathrm{op}}(b', b),
\end{tikzcd}$$

where σ is the symmetry of $\mathcal{V}$ and $M_{\mathcal{B}}$ and $M_{\mathcal{B}^{\mathrm{op}}}$ are the composition of the respective categories. The counit ϵ^F of $F_* \dashv F^*$ is the map induced by the top arrow as a varies, while the counit $\epsilon^{F^{\mathrm{op}}}$ of $(F^{\mathrm{op}})_* \dashv (F^{\mathrm{op}})^*$ is the map induced by the bottom arrow. Clearly, ϵ^F is an isomorphism iff $\epsilon^{F^{\mathrm{op}}}$ is. ☐

Since pseudofunctors preserve adjunctions, it is easy to say when a left adjoint is Cauchy dense. In fact, thanks to the self-duality of Cauchy density, the same applies to right adjoints:

Proposition 3.1.4. *A functor with an adjoint (either left or right) is Cauchy dense iff said adjoint is fully faithful.*

Proof. If $F\colon \mathcal{A} \to \mathcal{B}$ has a right adjoint G, then $F_* \dashv G_*$ and $F^* \cong G_*$, since adjoints are unique up to isomorphism. The counit of the profunctor adjunction $F_* \dashv G_*$ is given by ϵ_*, where ϵ is the counit of $F \dashv G$. This is an isomorphism iff ϵ is an isomorphism iff G is fully faithful. If F has a left adjoint, we apply this argument to F^{op} instead. ☐

This proposition gives yet another proof of the folklore fact that, if $H \dashv F \dashv G$ is an adjoint triple of functors, then H is fully faithful iff G is, for both conditions are equivalent to F being Cauchy dense.

Example 3.1.5. Any reflection and coreflection is Cauchy dense. Some examples are the free functors $\mathsf{Mon} \to \mathsf{Grp}$ and $\mathsf{Grp} \to \mathsf{Ab}$, the forgetful functor $\mathsf{Top} \to \mathsf{Set}$, and the completion functor from Met to its full subcategory on the complete metric spaces. These categories are not small, but one can make sense of these statements using the contents of Subsection 3.1.1 below.

Let us call a 1-cell of a 2-category **cofully faithful** if it is fully faithful as a 1-cell of $\mathcal{K}^{\mathrm{op}}$ (Definition 2.2.8). Explicitly, $f\colon x \to y$ is cofully faithful iff the ordinary functors $f^\circ\colon \mathcal{K}(y, z) \to \mathcal{K}(x, z)$ are fully faithful for each $z \in \mathcal{K}$. By the dual of Lemma 2.2.9, F is Cauchy dense iff F_* is cofully faithful as a 1-cell of $\mathcal{V}$-Prof. Since locally fully faithful pseudofunctors reflect (co)fully faithful 1-cells, it follows that F is cofully faithful as a 1-cell of $\mathcal{V}$-Cat. In fact, the converse is also true. The next theorem follows from the stronger Theorem 5.6 of Lucatelli Nunes and Sousa [47], although the proof here is independent.

Theorem 3.1.6 (Lucatelli Nunes and Sousa)**.** *A* 1*-cell of* $\mathcal{V}$-Cat *is cofully faithful iff it is a Cauchy dense functor.*

Proof. We have just seen the proof of the backwards implication in the previous paragraph. For the forwards one, suppose $F\colon \mathcal{A} \to \mathcal{B}$ is cofully faithful as a 1-cell of $\mathcal{V}$-Cat. We must show that F_* is cofully faithful as a 1-cell of $\mathcal{V}$-Prof. Given profunctors $\Phi, \Psi\colon \mathcal{B} \nrightarrow \mathcal{C}$, we have

$$\begin{aligned}
\mathcal{V}\text{-}\mathsf{Prof}(\Phi, \Psi) &\cong [\mathcal{C}^{\mathrm{op}} \otimes \mathcal{B}, \mathcal{V}]_0(\Phi, \Psi) \\
&\cong [\mathcal{B}, [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]]_0(\Phi, \Psi) \\
&\cong [\mathcal{A}, [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]]_0(\Phi \circ F, \Psi \circ F) \\
&\cong [\mathcal{C}^{\mathrm{op}} \otimes \mathcal{A}, \mathcal{V}]_0(\Phi(-, F-), \Psi(-, F-)) \\
&\cong \mathcal{V}\text{-}\mathsf{Prof}(\Phi(-, F-), \Psi(-, F-)) \\
&\cong \mathcal{V}\text{-}\mathsf{Prof}(\Phi \otimes F_*, \Psi \otimes F_*) && \text{by (2.1),}
\end{aligned}$$

where we have used Φ and Ψ to also denote the respective curried functors $\mathcal{B} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$. The third isomorphism follows from the fact that F is cofully faithful as a 1-cell of $\mathcal{V}$-Cat since, even though $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is not small, Φ and Ψ jointly factor through some small subcategory because $\mathcal{B}$ is small. ☐

Remark 3.1.7. The previous theorem can be interpreted as saying that the pseudofunctor $(-)_*\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$ not only reflects, but also preserves cofully faithful 1-cells. This is in stark contrast with the situation for fully faithful 1-cells. Recall from Lemma 2.2.10 that a fully faithful 1-cell of $\mathcal{V}$-Cat need not be a fully faithful $\mathcal{V}$-functor, but the latter condition is equivalent to F_* being fully faithful. This difference is explained by the absence of a formula like (2.1) for $F_* \otimes -$.

The isomorphism of sets

$$[\mathcal{B}, [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]]_0(\Phi, \Psi) \xrightarrow{\cong} [\mathcal{A}, [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]]_0(\Phi \circ F, \Psi \circ F)$$

in the proof of the previous theorem actually lifts to an isomorphism in $\mathcal{V}$. Indeed, the $\mathcal{V}$-functor $- \circ F\colon [\mathcal{B}, [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]] \to [\mathcal{A}, [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]]$ has a left adjoint Lan_F (since $\mathcal{A}$ is small and $[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$ is cocomplete). For such functors, the above isomorphism of sets is enough to ensure an isomorphism in $\mathcal{V}$, as shown by Kelly [33, p. 24]. Taking Φ and Ψ representable, it follows that F is Cauchy dense iff the $\mathcal{V}$-functor $[F, \mathcal{C}]\colon [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$ (and not just the underlying ordinary functor) is fully faithful for all small $\mathcal{C}$. This recovers some of the remaining parts of Theorem 5.6 of Lucatelli Nunes and Sousa [47].

Their other characterisation of Cauchy dense functors is as the absolutely (co)dense functors (Theorem 5.11 of the same article). Recall from Definition 2.2.11 that a functor $F\colon \mathcal{A} \to \mathcal{B}$ is dense if $\mathrm{Lan}_F F = 1_{\mathcal{B}}$. It is **absolutely dense** if, moreover, $\mathrm{Lan}_F F$ is an absolute left Kan extension. Similarly, F is **absolutely codense** if $\mathrm{Ran}_F F = 1_{\mathcal{B}}$ is an absolute Kan extension. Once again, we give a simplified proof here using the proarrow equipment $(-)_*$.

Theorem 3.1.8 (Lucatelli Nunes and Sousa). *The following are equivalent for a functor $F\colon \mathcal{A} \to \mathcal{B}$ between small categories:*

(i) *F is Cauchy dense;*

(ii) *F is absolutely codense;*

(iii) *F is absolutely dense.*

Proof. Let F be Cauchy dense and $G\colon \mathcal{B} \to \mathcal{C}$. Then

$$\begin{aligned} \operatorname{ran}_{F_*}(G_* \otimes F_*) &\cong \operatorname{ran}_{F_*}(\operatorname{ran}_{F^*} G_*) && \text{by Lemma 2.2.3(iii)} \\ &\cong \operatorname{ran}_{F_* \otimes F^*} G_* && \text{by Lemma 2.2.3(iv)} \\ &\cong G_* && \text{since } F_* \otimes F^* \cong 1. \end{aligned}$$

Since all extensions of functors are understood to be pointwise and these are exactly the ones preserved by $(-)_*$, this shows that F is absolutely codense.

Now suppose F is absolutely codense. Then $F^{\mathrm{op}}\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}}$ is absolutely dense. In particular, $\operatorname{Lan}_{F^{\mathrm{op}}} \mathcal{B}(F-, b) \cong \mathcal{B}(-, b)$ for any $b \in \mathcal{B}$. But this left Kan extension is precisely $(F_* \otimes F^*)(-, b)$ by (2.6) and this isomorphism is the counit of $F_* \dashv F^*$. Hence, F is Cauchy dense.

The proof that (i) and (iii) are equivalent is dual by Proposition 3.1.3. □

Corollary 3.1.9. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be left adjoint to G. Then the following conditions are equivalent:*

(a) *G is fully faithful;*

(b) *F is dense;*

(c) *F is absolutely dense;*

(d) *F is absolutely codense;*

(e) *F is Cauchy dense.*

Proof. The counit of the Kan extension $\operatorname{Ran}_F F = FG \to 1_{\mathcal{B}}$ is the counit of the adjunction by Lemma 2.2.3(iii), so (a) and (b) are equivalent. The remaining conditions are equivalent to (a) by Proposition 3.1.4 and Theorem 3.1.8. □

Unlike (co)dense functors, Cauchy dense functors are closed under composition. Parts (a) and (c) of the next proposition also follow from the characterisation of Cauchy dense functors as the cofully faithful 1-cells of $\mathcal{V}$-**Cat**.

Proposition 3.1.10. *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be functors between small categories.*

(a) *If F and G are Cauchy dense, then so is GF.*

(b) *If GF is Cauchy dense and G is fully faithful, then F is Cauchy dense.*

(c) *If GF and F are Cauchy dense, then so is G.*

Proof. The counit ϵ^{GF} of the profunctor adjunction $(GF)_* \dashv (GF)^*$ is the composite

$$G_* \otimes F_* \otimes F^* \otimes G^* \xrightarrow{G_* \otimes \epsilon^F \otimes G^*} G_* \otimes G^* \xrightarrow{\epsilon^G} 1_{\mathcal{C}},$$

where ϵ^F and ϵ^G are the counits of $F_* \dashv F^*$ and $G_* \dashv G^*$, respectively.

If ϵ^F and ϵ^G are isomorphisms, then so is ϵ^{GF}, giving (a). Similarly, if ϵ^F and ϵ^{GF} are isomorphisms, then so is ϵ^G, giving (c).

For (b), note that for all $b, b' \in \mathcal{B}$ we have a commutative square

$$\begin{tikzcd}
\int^a \mathcal{B}(Fa, b') \otimes \mathcal{B}(b, Fa) \arrow[r, "\epsilon^F_{b,b'}"] \arrow[d, "\int^a G \otimes G"'] & \mathcal{B}(b, b') \arrow[d, "G"] \\
\int^a \mathcal{B}(GFa, Gb') \otimes \mathcal{B}(Gb, GFa) \arrow[r, "\epsilon^{GF}_{Gb,Gb'}"'] & \mathcal{C}(Gb, Gb').
\end{tikzcd}$$

If G is fully faithful, then the vertical arrows are isomorphisms. Hence, if ϵ^{GF} is an isomorphism, so is ϵ^F. ☐

The previous proposition shows that fully faithful Cauchy dense functors enjoy the two-out-of-three property. Such functors will be the subject of Section 3.2.

Remark 3.1.11. If one removes from (b) the assumption that G is fully faithful, then the result is no longer true, even if full faithfulness is replaced by Cauchy density. An example can be easily found among functors between groups, since for such functors Cauchy density is equivalent to surjectivity (i.e. fullness) by Corollary 3.3.18 below.

If one removes from (c) the assumption that F is Cauchy dense, then the result is no longer true. A counterexample can easily be found among functors between discrete categories, since for such functors Cauchy density is equivalent to bijectivity (on objects) by Corollary 3.3.19 below.

Any lax monoidal functor $S\colon \mathcal{V}_0 \to \mathcal{W}_0$ gives a 2-functor $S_*\colon \mathcal{V}\text{-}\mathsf{CAT} \to \mathcal{W}\text{-}\mathsf{CAT}$, often called **base change**. The next proposition gives some elementary results on how Cauchy density behaves under base change when S is strong monoidal.

Proposition 3.1.12. *Let $S\colon \mathcal{V}_0 \to \mathcal{W}_0$ be a strong monoidal functor, and $F\colon \mathcal{A} \to \mathcal{B}$ be a $\mathcal{V}$-functor.*

(a) *If S is cocontinuous and F is Cauchy dense, then so is S_*F.*

(b) *If S reflects colimits and S_*F is Cauchy dense, then so is F.*

Proof. For each $b, b' \in \mathcal{B}$ and $a \in \mathcal{A}$, we have a commutative triangle in $\mathcal{W}_0$

$$\begin{tikzcd}
S\mathcal{B}(Fa, b') \otimes S\mathcal{B}(b, Fa) \arrow[d, "\mu"', "\cong"] \arrow[dr, "M_{S_*\mathcal{B}}"] & \\
S(\mathcal{B}(Fa, b') \otimes \mathcal{B}(b, Fa)) \arrow[r, "SM_{\mathcal{B}}"'] & S\mathcal{B}(b, b'),
\end{tikzcd}$$

where μ is the coherence isomorphism of S. For (a), the cowedge $M_{\mathcal{B}}$ is a coend cowedge. Since S is cocontinuous, so is $SM_{\mathcal{B}}$ and $M_{S_*\mathcal{B}}$. For (b), the cowedge $M_{S_*\mathcal{B}}$ is a coend cowedge, and then so is $SM_{\mathcal{B}}$. Since S reflects colimits, $M_{\mathcal{B}}$ is also a coend cowedge. ☐

Example 3.1.13. Let $\Omega = \{\bot < \top\}$ denote the walking arrow category. The representable functor $\Omega(\top, -) : \Omega \to \mathsf{Set}$ is readily seen to be strong monoidal. Moreover, it reflects colimits, being fully faithful, so an order preserving map $f\colon A \to B$ between two Ω-enriched categories is Cauchy dense if the corresponding functor $\Omega(\top, -)_* f$ between preorders is Cauchy dense. The converse is not true; see Subsection 3.3.2.

3.1.1 Between large categories

We conclude this section with some generalisations to large categories. For $\mathcal{A}$ and $\mathcal{B}$ two not-necessarily-small categories, we say that $F\colon \mathcal{A} \to \mathcal{B}$ is **Cauchy dense** if the coend

$$\int^{a} \mathcal{B}(Fa, b') \otimes \mathcal{B}(b, Fa)$$

exists and the map from it to $\mathcal{B}(b, b')$ induced by the composition of $\mathcal{B}$ is an isomorphism for all $b, b' \in \mathcal{B}$. Of course, this agrees with Definition 3.1.1 when $\mathcal{A}$ and $\mathcal{B}$ are small, since $\mathcal{V}$ is cocomplete.

The proof of Proposition 3.1.3 still applies to this new definition, so that F is Cauchy dense iff F^{op} is. Note also that this coend is the pointwise formula for $\mathrm{Lan}_F\, \mathcal{B}(b, F-)$ evaluated at b', so that its existence for all b' gives the pointwise existence of this left Kan extension.

In general, when $\mathcal{A}$ is large, there is no functor $\mathcal{V}$-category $[\mathcal{A}, \mathcal{B}]$, because defining the $\mathcal{V}$-object of $\mathcal{V}$-natural transformations between two functors will involve a large end. However, one can still generalise the parts of Theorem 3.1.8 that do not involve functor categories.

Theorem 3.1.14. *The following are equivalent for a functor $F\colon \mathcal{A} \to \mathcal{B}$ between (not necessarily small) categories:*

(a) *F is Cauchy dense;*

(b) *F is absolutely dense;*

(c) *F is absolutely codense.*

Proof. We show that (a) and (c) are equivalent. The equivalence of (a) and (b) will then follow from the fact that Cauchy density is a self-dual condition.

Let F be Cauchy dense and $H\colon \mathcal{B} \to \mathcal{C}$ be an arbitrary functor. We must show that Hb satisfies the universal property of $\mathrm{Ran}_F\, HF$ evaluated at b for all $b \in \mathcal{B}$. Let $c \in \mathcal{C}$. We have a series of natural isomorphisms

$$\begin{aligned} \mathcal{C}(c, Hb) &\cong [\mathcal{B}, \mathcal{V}](\mathcal{B}(b, -), \mathcal{C}(c, H-)) && \text{by Yoneda,} \\ &\cong [\mathcal{B}, \mathcal{V}](\mathrm{Lan}_F\, \mathcal{B}(b, F-), \mathcal{C}(c, H-)) && \text{since } F \text{ is Cauchy dense,} \\ &\cong [\mathcal{A}, \mathcal{V}](\mathcal{B}(b, F-), \mathcal{C}(c, HF-)) && \text{by the universal property of Lan.} \end{aligned}$$

Note that, a priori, only the first object of $\mathcal{V}$ listed exists, since the rest require some weighted limit in $\mathcal{V}$ indexed by a large category. However, the series of isomorphisms shows that, in fact, all of them exist. This proves that Hb satisfies the defining universal property of $\{\mathcal{B}(b, F-), HF\}$, which gives the value at b of $\mathrm{Ran}_F HF$.

Conversely, let F be absolutely codense. Then F^{op} is absolutely dense, which implies that $\mathrm{Lan}_{F^{\mathrm{op}}} \mathcal{B}(F-, b') = \mathcal{B}(-, b')$ for all $b' \in \mathcal{B}$. This is equivalent to saying that F is Cauchy dense. □

With this, Proposition 3.1.4 and Corollary 3.1.9 remain true for functors between large categories. This justifies the claims made in Example 3.1.5, since if F has a left or right adjoint the relevant coends always exists.

Remark 3.1.15. With this definition of Cauchy density of functors between large categories, one can try to remove the hypothesis in Proposition 3.1.10 that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are small. Part (b) is still true, with the same proof. Parts (a) and (c) hold as soon as $\mathcal{A}$ and $\mathcal{B}$ are small.

3.2 Fully faithful Cauchy dense functors

Given a metric space A, its completion $\overline{A}$ is the largest space into which A embeds as a dense subset. In our language, this means that if $f\colon A \to B$ is a fully faithful Cauchy dense $\mathbb{R}^+$-functor, then B embeds into $\overline{A}$. The main purpose of this section is to generalise this fact to $\mathcal{V}$-categories.

Given a small category $\mathcal{A}$ we write $\overline{\mathcal{A}}$ for its **Cauchy completion**. It is the full subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ on those presheaves that satisfy any of the equivalent conditions in the next theorem, a proof of which can be assembled from the contents of Kelly and Schmitt [35, §6], where they also cite Street [59].

Theorem 3.2.1 (Kelly and Schmitt, Street)**.** *Let $\mathcal{A}$ be a small category. For a functor $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$, the following are equivalent:*

(a) *W is **small-projective**, i.e. $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](W, -)$ preserves all small colimits;*

(b) *W is an **absolute weight**, i.e. W-weighted colimits are preserved by any functor;*

(c) *W has a right adjoint in $\mathcal{V}$-Prof when seen as profunctor $1 \nrightarrow \mathcal{A}$.*

The category $\mathcal{A}$ is **Cauchy complete** if $\overline{\mathcal{A}}$ coincides with $\mathcal{A}$, or rather with its image under the Yoneda embedding. Equivalently, $\mathcal{A}$ is Cauchy complete if it has all colimits weighted by absolute weights. The name 'Cauchy completion' was suggested by Lawvere [43, §3], motivated by the fact that when A is a metric space, its Cauchy completion $\overline{A}$ in the $\mathbb{R}^+$-enriched sense can be identified with the familiar metric completion of A (the space of equivalence classes of Cauchy sequences in A).

Our main theorem relates fully faithful Cauchy dense functors and Cauchy completions. The key result that allows this, Lemma 3.2.3 below, actually holds in slightly more generality for split-full Cauchy dense functors.

Definition 3.2.2. A functor $F\colon \mathcal{A} \to \mathcal{B}$ is **split-full** if

$$F_{a,a'}\colon \mathcal{A}(a,a') \longrightarrow \mathcal{B}(Fa,Fa')$$

is a split epimorphism for all $a, a' \in \mathcal{A}$.

Assuming the axiom of choice, an ordinary functor is split-full iff it is full. For different $\mathcal{V}$, however, this is stronger than merely requiring $F_{a,a'}$ to be epic. For example, if $\mathcal{V}_0$ is a preorder, then a $\mathcal{V}$-functor is split-full iff it is fully faithful.

Lemma 3.2.3. *Let $\mathcal{A}$ be a small category and $F\colon \mathcal{A} \to \mathcal{B}$ be a split-full Cauchy dense functor. Then $\mathcal{B}(F-,b)\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ lies in $\overline{\mathcal{A}}$ for all $b \in \mathcal{B}$.*

Proof. We will show that $\mathcal{B}(F-,b)\colon \mathbf{1} \nrightarrow \mathcal{A}$ is left adjoint to $\mathcal{B}(b,F-)\colon \mathcal{A} \nrightarrow \mathbf{1}$ in $\mathcal{V}$-Prof. The unit is a morphism $I \to \int^a \mathcal{B}(Fa,b) \otimes \mathcal{B}(b,Fa)$ in $\mathcal{V}$, which we take to be the composite

$$\delta_* = \; I \xrightarrow{j_b} \mathcal{B}(b,b) \xrightarrow{\epsilon_{b,b}^{-1}} \int^a \mathcal{B}(Fa,b) \otimes \mathcal{B}(b,Fa).$$

The counit has a component $\nu_{a,a'}\colon \mathcal{B}(b,Fa') \otimes \mathcal{B}(Fa,b) \to \mathcal{A}(a,a')$ for each $a, a' \in \mathcal{A}$, which we take to be the composite

$$\nu_{a,a'} = \; \mathcal{B}(b,Fa') \otimes \mathcal{B}(Fa,b) \xrightarrow{M_{\mathcal{B}}} \mathcal{B}(Fa,Fa') \xrightarrow{F^r_{a,a'}} \mathcal{A}(a,a'),$$

where $F^r_{a,a'}$ is a right inverse to $F_{a,a'}$.

We check one of the triangle identities, the other being exactly dual. For each $a \in \mathcal{A}$, consider the following diagram in $\mathcal{V}$:

$$\begin{tikzcd}
I \otimes \mathcal{B}(Fa,b) \arrow[r, "j_b \otimes 1"] \arrow[dr, "l"'] & \mathcal{B}(b,b) \otimes \mathcal{B}(Fa,b) \arrow[r, "\epsilon_{b,b}^{-1} \otimes 1"] \arrow[d, "M_{\mathcal{B}}"] & \int^{a'} \mathcal{B}(Fa',b) \otimes \mathcal{B}(b,Fa') \otimes \mathcal{B}(Fa,b) \arrow[d, "\int 1 \otimes M_{\mathcal{B}}"] \\
 & \mathcal{B}(Fa,b) \arrow[dr, "i^{-1}"'] & \int^{a'} \mathcal{B}(Fa',b) \otimes \mathcal{B}(Fa,Fa') \arrow[l, "\epsilon_{Fa,b}"'] \arrow[d, "\int 1 \otimes F^r_{a,a'}"] \\
 & & \int^{a'} \mathcal{B}(Fa',b) \otimes \mathcal{A}(a,a')
\end{tikzcd}$$

The arrow labelled l is the left unitor in $\mathcal{V}$, and i is the isomorphism in the density formula.

The triangle on the left commutes by unitality in $\mathcal{B}$. The square commutes since the same square with the top arrow replaced by its inverse commutes, by associativity in $\mathcal{B}$.

It remains to check the commutativity of the bottom triangle. The density formula implies that $\epsilon_{Fa,b} \cdot (\int 1 \otimes F_{a,a'}) = i$, where we use $\cdot$ to denote composition in $\mathcal{V}_0$. It follows that $i^{-1} \cdot \epsilon_{Fa,b} = \int 1 \otimes F^r_{a,a'}$, as needed.

This shows that $(\nu \otimes \mathcal{B}(F-, b)) \cdot (\mathcal{B}(F-, b) \otimes \delta)$ is the identity on $\mathcal{B}(F-, b)$, thus establishing one of the triangle identities. □

It is no coincidence that in the previous proof we have only used that $\epsilon_{b,b}$ is invertible for all $b \in \mathcal{B}$, rather than the full strength of Cauchy density, i.e. that $\epsilon_{b,b'}$ is invertible for all $b, b' \in \mathcal{B}$. For split-full functors, Cauchy density is equivalent to this apparently weaker condition (Theorem 3.3.20).

By the characterisation of free cocompletions under classes of diagrams in Theorem 2.2.15, the previous lemma shows that a fully faithful Cauchy dense functor amounts to the inclusion of a category $\mathcal{A}$ into its free cocompletion under a class of diagrams whose weights are absolute.

Remark 3.2.4. If one strengthens the hypotheses of Lemma 3.2.3 so that F is fully faithful and Cauchy dense, then a simpler proof is possible through Theorem 3.1.14, Avery and Leinster's [8, Lem. 8.1] and Kelly and Schmitt's [35, Prop. 6.14]. In fact, from our lemma and the latter result, it follows that if F is split-full and Cauchy dense, then $\mathcal{B}(F-, b)$ and $\mathcal{B}(b, F-)$ are a pair of Isbell conjugate presheaves.

We are now ready to prove the universal property of the Cauchy completion with respect to fully faithful Cauchy dense functors.

Theorem 3.2.5. *Let $\mathcal{A}$ be a small category. Then $\overline{\mathcal{A}}$ is the largest category that admits a fully faithful Cauchy dense functor from $\mathcal{A}$, i.e. for every fully faithful Cauchy dense functor $F\colon \mathcal{A} \to \mathcal{B}$ there exists an essentially unique functor (necessarily fully faithful) N_F such that the triangle*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, hook, "z"] \arrow[dr, hook, "F"'] & & \overline{\mathcal{A}} \\
& \mathcal{B} \arrow[ur, hook, "N_F"'] &
\end{tikzcd}$$

commutes up to isomorphism, where z is the factorisation of the Yoneda embedding. Moreover, the inclusion of $\mathcal{A}$ into any full subcategory of $\overline{\mathcal{A}}$ containing the representables is Cauchy dense.

Proof. We take $N_F(b) = \mathcal{B}(F-, b)$, which by Lemma 3.2.3 lies in $\overline{\mathcal{A}}$. Then $N_F(Fa) = \mathcal{B}(F-, Fa) \cong \mathcal{A}(-, a) = za$ naturally in $a \in \mathcal{A}$ because F is fully faithful, so the triangle in the statement commutes up to isomorphism. By Theorem 3.1.8, F is dense, which is to say that N_F is fully faithful.

Since N_F reflects colimits, each object of $\mathcal{B}$ is an absolute colimit of objects in the image of F. Any other functor $N'_F\colon \mathcal{B} \to \overline{\mathcal{A}}$ completing the triangle in the statement preserves these colimits, and hence must be isomorphic to N_F.

For the last statement, it suffices by Proposition 3.1.10(b) and Remark 3.1.15 to show that z is Cauchy dense. It is so, by Theorem 3.1.14, because it is absolutely dense by the definition of $\overline{\mathcal{A}}$. □

Avery and Leinster showed in [8, Cor. 8.3 and Thm. 8.4] that the reflexive completion $\mathcal{R}(\mathcal{A})$ of a small category $\mathcal{A}$ is the largest category into which $\mathcal{A}$

admits an adequate functor, i.e. a functor $F\colon \mathcal{A} \to \mathcal{B}$ which is fully faithful, dense and codense. Theorem 3.2.5 shows that the Cauchy completion admits a similar description, where adequate functors are replaced by fully faithful Cauchy dense functors. Since every Cauchy dense functor is dense and codense, it follows that $\overline{\mathcal{A}} \subseteq \mathcal{R}(\mathcal{A})$, as Avery and Leinster have already shown [8, Prop. 10.1].

Corollary 3.2.6. *Two small categories $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, i.e. $\overline{\mathcal{A}} \simeq \overline{\mathcal{B}}$, iff there is a zigzag of fully faithful Cauchy dense functors connecting them.*

Proof. If $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then $\mathcal{A} \xhookrightarrow{z_{\mathcal{A}}} \overline{\mathcal{A}} \simeq \overline{\mathcal{B}} \xhookleftarrow{z_{\mathcal{B}}} \mathcal{B}$ gives the desired zigzag.

Conversely, it suffices to show that if $F\colon \mathcal{A} \to \mathcal{B}$ is fully faithful and Cauchy dense, then $\overline{\mathcal{A}} \simeq \overline{\mathcal{B}}$. Consider the following diagram of fully faithful functors:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook, "z_{\mathcal{A}}"] \arrow[d, "F"'] & \overline{\mathcal{A}} \arrow[d, hook, "N_{N_F}"] \\
\mathcal{B} \arrow[r, hook, "z_{\mathcal{B}}"] \arrow[ur, hook, "N_F"] & \overline{\mathcal{B}}. \arrow[u, hook, bend right=70, "N_{z_{\mathcal{B}}F}"']
\end{tikzcd}$$

Note that N_F and $z_{\mathcal{B}}F$ are fully faithful and Cauchy dense by Proposition 3.1.10(b) and Remark 3.1.15, so N_{N_F} and $N_{z_{\mathcal{B}}F}$ exist by Theorem 3.2.5. Then $N_{z_{\mathcal{B}}F}N_{N_F}$ and $N_{N_F}N_{z_{\mathcal{B}}F}$ are isomorphic to the respective identities, because they are so on representables and Cauchy completion is the free cocompletion with respect to absolute colimits. ☐

The next theorem gives an alternative way of identifying the fully faithful Cauchy dense functors among the Cauchy dense ones, in view of Theorem 3.1.8.

Theorem 3.2.7. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between small categories. Then F is fully faithful and Cauchy dense iff the functor $[F, \mathcal{C}]\colon [\mathcal{B}, \mathcal{C}] \to [\mathcal{A}, \mathcal{C}]$ is an equivalence for every Cauchy complete category $\mathcal{C}$.*

Proof. First assume that F is fully faithful and Cauchy dense. In the commutative diagram

$$\begin{tikzcd}
{[\overline{\mathcal{B}}, \mathcal{C}]} \arrow[r, "{[z_{\mathcal{B}}, \mathcal{C}]}"] \arrow[d, "{[N_{N_F}, \mathcal{C}]}"'] & {[\mathcal{B}, \mathcal{C}]} \arrow[d, "{[F, \mathcal{C}]}"] \\
{[\overline{\mathcal{A}}, \mathcal{C}]} \arrow[r, "{[z_{\mathcal{A}}, \mathcal{C}]}"'] & {[\mathcal{A}, \mathcal{C}],}
\end{tikzcd}$$

the horizontal arrows are equivalences by Kelly and Schmitt's [35, 7.1], and so is the left one by the proof of Corollary 3.2.6. Therefore, $[F, \mathcal{C}]$ must be an equivalence too.

Conversely, consider the commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "y_{\mathcal{A}}"] \arrow[d, "F"'] & {[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]} \arrow[d, "\widehat{F}"] \\
\mathcal{B} \arrow[r, "y_{\mathcal{B}}"'] & {[\mathcal{B}^{\mathrm{op}}, \mathcal{V}],}
\end{tikzcd}$$

where $\widehat{F} = \mathrm{Lan}_{y_{\mathcal{A}}} y_{\mathcal{B}} F$ is the unique-up-to-isomorphism cocontinuous functor that makes the diagram commute. In such cases, $\widehat{F}$ always has a right adjoint, which sends $P \colon \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ to $[\mathcal{B}^{\mathrm{op}}, \mathcal{V}](y_{\mathcal{B}}F-, P)$ (see e.g. Kelly's [33, Thm 4.51]). By Yoneda, this right adjoint is just $[F^{\mathrm{op}}, \mathcal{V}]$. Since $\mathcal{V}$ is complete, $\mathcal{V}^{\mathrm{op}}$ is Cauchy complete, so $[F, \mathcal{V}^{\mathrm{op}}]$ is an equivalence, and then so is $[F, \mathcal{V}^{\mathrm{op}}]^{\mathrm{op}} = [F^{\mathrm{op}}, \mathcal{V}]$. Its left adjoint, $\widehat{F}$, must also be an equivalence. It follows that F is fully faithful.

Moreover, $\widehat{F}$ restricts to an equivalence between the full-subcategories of small-projective objects, i.e. an equivalence $\overline{F} \colon \overline{\mathcal{A}} \simeq \overline{\mathcal{B}}$. Since $\overline{F} z_{\mathcal{A}} \cong z_{\mathcal{B}} F$ is Cauchy dense and $z_{\mathcal{B}}$ is fully faithful, Proposition 3.1.10(b) gives that F is Cauchy dense. ☐

A closely related characterisation of fully faithful Cauchy dense functors appears in work of Lucatelli Nunes, Prezado and Sousa [53, Thm. 2.5]. The author was unaware of their result at the time of writing.

3.3 Special cases

This last section deals with special cases and characterisations of Cauchy dense functors in different contexts.

3.3.1 When $\mathcal{V}_0$ is a preorder

In Proposition 3.1.2, we saw that for metric spaces Cauchy density reduces to topological density. This condition quantifies over each point of the codomain, instead of over each pair of points of the codomain, as Cauchy density does. If $\mathcal{V}_0$ is a preorder, it turns out that one can always make this simplification.

Proposition 3.3.1. *Let $\mathcal{V}_0$ be a preorder. A $\mathcal{V}$-functor $F \colon \mathcal{A} \to \mathcal{B}$ is Cauchy dense iff $\epsilon_{b,b} \colon (F_* \otimes F^*)(b,b) \to \mathcal{B}(b,b)$ is an isomorphism for all $b \in \mathcal{B}$. Moreover, whether F is Cauchy dense only depends on the image of its object mapping.*

Proof. The forward implication is clear. Now assume that $\epsilon_{b,b}$ is an isomorphism for all $b \in \mathcal{B}$. Let $b, b' \in \mathcal{B}$ and consider the following diagram in $\mathcal{V}_0$

$$\begin{tikzcd}
\int^a \mathcal{B}(Fa,b') \otimes \mathcal{B}(b,Fa) \arrow[r, "\epsilon_{b,b'}"] & \mathcal{B}(b,b') \arrow[d, "1 \otimes j_b"] \\
\int^a \mathcal{B}(b,b') \otimes \mathcal{B}(Fa,b) \otimes \mathcal{B}(b,Fa) \arrow[u, "\int^a M_{\mathcal{B}} \otimes 1"] \arrow[r, "1 \otimes \epsilon_{b,b}"'] & \mathcal{B}(b,b') \otimes \mathcal{B}(b,b).
\end{tikzcd}$$

By assumption, the bottom arrow has an inverse, so we can compose with it to get a morphism in the opposite direction to $\epsilon_{b,b'}$. Since $\mathcal{V}_0$ is a preorder, this suffices to show that $\epsilon_{b,b'}$ is an isomorphism.

For the last statement, note that since $\mathcal{V}_0$ is a preorder, coends are simply suprema. Hence, whether $\epsilon_{b,b'}$ is an isomorphism only depends on the action of F on objects. ☐

Remark 3.3.2. This is manifestly not true when $\mathcal{V}_0$ is not a preorder. The diagram in the previous proof does not typically commute. For example, the identity-on-objects functor from the discrete category $\{\bot, \top\}$ to the preorder $\Omega = \{\bot < \top\}$ has $\epsilon_{b,b}$ an isomorphism for all $b \in \Omega$, but is not Cauchy dense. Nevertheless, there is a similar result for general $\mathcal{V}$ if one assumes that F is split-full. This is the content of Subsection 3.3.5.

3.3.2 Between preorders

Let Ω be as in the previous remark, i.e. the walking arrow category, which is cartesian closed. An Ω-category is a preorder, and an Ω-functor $f \colon A \to B$ is an order preserving map.

Proposition 3.3.3. *An Ω-functor $f \colon A \to B$ is Cauchy dense iff it is essentially surjective.*

Proof. Using Proposition 3.3.1, the condition that f be Cauchy dense translates in this context to

$$\bigvee_a [(fa \leq b) \wedge (b \leq fa)] = \top, \qquad \forall b \in B.$$

This is equivalent to there being some $a \in A$ such that $fa \cong b$, in the sense that $fa \leq b$ and $b \leq fa$. □

Of course, one can also view preorders as ordinary categories, instead of Ω-categories. This is thanks to the strong monoidal functor $\Omega(\top, -) \colon \Omega \to \mathsf{Set}$, which gives an inclusion $\Omega\text{-}\mathsf{Cat} \hookrightarrow \mathsf{Cat}$. For ordinary functors between preorders in Cat, Cauchy density is a stronger condition than for Ω-functors.

Proposition 3.3.4. *An ordinary functor $f \colon A \to B$ between preorders is Cauchy dense iff*

$$\pi_0\{a \in A \mid b \leq fa \leq b'\} = \begin{cases} 1 & \textit{if } b \leq b', \\ \varnothing & \textit{otherwise.} \end{cases}$$

Proof. Using the computation of coends in Set, we see that f is Cauchy dense iff for all $b, b' \in B$ the obvious diagram

$$\begin{tikzcd} \coprod\limits_{a \leq a'} B(fa', b') \times B(b, fa) \arrow[r, shift left] \arrow[r, shift right] & \coprod\limits_{a} B(fa, b') \times B(b, fa) \arrow[r] & B(b, b'), \end{tikzcd}$$

is a coequaliser diagram. Now consider the preorder (really, a double comma category) $b/f/b' = \{a \in A \mid b \leq fa \leq b'\} \subseteq A$ in the statement. The above coincides with the computation using coproducts and coequalisers of the colimit of the functor $b/f/b' \to \mathsf{Set}$ constant at 1, which gives precisely the required connected components. □

It follows that Cauchy density is not preserved under base change in general.

3.3.3 Between monoids in $\mathcal{V}$

In this subsection we study the behaviour of Cauchy dense functors between one-object $\mathcal{V}$-categories, i.e. between monoids in $\mathcal{V}$. We will temporarily relax our assumptions: we only require that $\mathcal{V}$ be a monoidal category with reflexive coequalisers, preserved by the monoidal product in each variable.

When specialised to monoids, the theory of profunctors becomes the familiar theory of bimodules. For $A, B \in \mathcal{V}\text{-}\mathsf{Mon}$, a profunctor $M\colon A \nrightarrow B$ amounts to an (A, B)-bimodule. The composite of M with $N\colon B \nrightarrow C$ is the tensor product $M \otimes_B N\colon A \nrightarrow C$ which is given by the coequaliser in $\mathcal{V}$ of the pair

$$\begin{tikzcd} M \otimes B \otimes N \arrow[r, shift left, "\rho^M \otimes 1"] \arrow[r, shift right, "1 \otimes \lambda^N"'] & M \otimes N, \end{tikzcd} \tag{3.1}$$

where ρ^M and λ^N are the right and left actions of B on M and N, respectively. This coequaliser exists under our assumptions on $\mathcal{V}$ because this pair is reflexive: a common section is constructed using the unit $\eta^B : I \to B$. The tensor product $M \otimes_B N$ has a left A-action and right C-action because $A \otimes -$ and $- \otimes C$ preserve this coequaliser. The notation $M \otimes_B N$ is borrowed from the $\mathcal{V} = \mathsf{Ab}$ case, where this is the familiar tensor product of the right B-module M with the left B-module N. The (ordinary) category of (B, B)-bimodules becomes monoidal with this tensor product, with the unitors corresponding to the isomorphisms of the density formula.

Let $f\colon A \to B$ be a morphism in $\mathcal{V}\text{-}\mathsf{Mon}$, seen as a functor between one-object $\mathcal{V}$-categories. The coend in Definition 3.1.1 then amounts to the tensor product $B \otimes_A B$, i.e. the coequaliser of

$$\begin{tikzcd} B \otimes A \otimes B \arrow[r, "1 \otimes f \otimes 1"] & B \otimes B \otimes B \arrow[r, shift left, "\mu^B \otimes 1"] \arrow[r, shift right, "1 \otimes \mu^B"'] & B \otimes B. \end{tikzcd}$$

Since B has an associative multiplication, μ^B coequalises this pair, and hence induces a unique morphism $\widehat{\mu}^B\colon B \otimes_A B \to B$ in $\mathcal{V}$, which actually lifts to a morphism of (B, B)-bimodules. With this notation, f is Cauchy dense iff $\widehat{\mu}^B$ is an isomorphism. Note that $\widehat{\mu}^B$ is always split epic: it has two canonical sections given by the composites of the commutative pentagons

$$\begin{tikzcd} B \arrow[r, "r^{-1}"] \arrow[d, "1 \otimes \eta^A"'] & B \otimes_A A \arrow[dd, "1 \otimes f"] \\ B \otimes A \arrow[d, "1 \otimes f"'] & \\ B \otimes B \arrow[r, two heads] & B \otimes_A B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} B \arrow[r, "l^{-1}"] \arrow[d, "\eta^A \otimes 1"'] & A \otimes_A B \arrow[dd, "f \otimes 1"] \\ A \otimes B \arrow[d, "f \otimes 1"'] & \\ B \otimes B \arrow[r, two heads] & B \otimes_A B, \end{tikzcd} \tag{3.2}$$

where $r\colon B \otimes_A A \to B$ and $l\colon A \otimes_A B \to B$ are the left and right unitors in the category of (A, A)-bimodules.

Lemma 3.3.5. *The following are equivalent for a morphism $f\colon A \to B$ in $\mathcal{V}$-Mon:*

(a) *f is Cauchy dense;*

(b) *the morphism $\widehat{\mu}^B\colon B \otimes_A B \to B$ induced by μ^B is an isomorphism;*

(c) *the two sections $(1 \otimes f) \cdot r^{-1}$ and $(f \otimes 1) \cdot l^{-1}$ in (3.2) coincide.*

Moreover, if these conditions hold, then f is an epimorphism in $\mathcal{V}$-Mon. The converse is true, at least when $\mathcal{V}$ is symmetric monoidal, A is commutative and f is in the centre of B, i.e.

$$\begin{tikzcd}
A \otimes B \arrow[rr, "\sigma"] \arrow[d, "f \otimes 1"'] & & B \otimes A \arrow[d, "1 \otimes f"] \\
B \otimes B \arrow[r, "\mu^B"'] & B & B \otimes B \arrow[l, "\mu^B"]
\end{tikzcd}$$

commutes, where $\sigma : A \otimes B \to B \otimes A$ is the symmetry.

Proof. That (a) is equivalent to (b) follows from the fact that $\widehat{\mu}^B$ is the unique leg of ϵ in Definition 3.1.1. Writing $\widehat{\eta}^B = f\colon A \to B$ for the unit of B as a monoid in the category of (A, A)-bimodules, we see that the two sections in (c) are exactly $B \otimes_A \widehat{\eta}^B$ and $\widehat{\eta}^B \otimes_A B$ respectively. The equivalence of (b) and (c) is then a general fact about monoids in a monoidal category.

To show that f is epic, let $g, h\colon B \to C$ be morphisms in $\mathcal{V}$-Mon such that $gf = hf$. This composite gives C a single (A, A)-bimodule structure, and g and h become maps of (A, A)-bimodules. Let $\widehat{\mu}^C : C \otimes_A C \to C$ be the morphism induced by μ^C. Since g is a monoid map, $g \cdot \widehat{\mu}^B = \widehat{\mu}^C \cdot (g \otimes g)$. This gives

$$\begin{aligned}
g &= g \cdot \widehat{\mu}^B \cdot (1 \otimes f) \cdot r^{-1} \\
&= \widehat{\mu}^C \cdot (g \otimes g) \cdot (1 \otimes f) \cdot r^{-1} \\
&= \widehat{\mu}^C \cdot (g \otimes gf) \cdot r^{-1} \\
&= \widehat{\mu}^C \cdot (g \otimes hf) \cdot r^{-1} \\
&= \widehat{\mu}^C \cdot (g \otimes h) \cdot (1 \otimes f) \cdot r^{-1}.
\end{aligned}$$

Similarly, we have $h = \widehat{\mu}^C \cdot (g \otimes h) \cdot (f \otimes 1) \cdot l^{-1}$. If (c) holds, then we must have $g = h$, as needed.

Lastly, if $\mathcal{V}$ is symmetric monoidal, A is commutative and f is in the centre of B, then $B \otimes_A B$ is actually a monoid, and the two maps whose equality is asserted in (c) are monoid maps. Moreover, f equalises them, so that if f is an epimorphism in $\mathcal{V}$-Mon then (c) holds. □

Remark 3.3.6. That Cauchy dense maps of monoids are epimorphisms does not immediately follow from the fact that Cauchy dense ordinary functors are cofully faithful in $\mathcal{V}$-Cat. For example, for $\mathcal{V} = \mathsf{Set}$, that only ensures that if $g, h\colon B \rightrightarrows C \in \mathsf{Mon}$ are such that $gf = hf$, then there is an invertible $u \in C$ such that $ug(b) = h(b)u$ for all $b \in B$.

Corollary 3.3.7. *If $\mathcal{V}$ is symmetric monoidal, the epimorphisms in $\mathcal{V}$-CMon are precisely the Cauchy dense morphisms.*

It turns out that for $\mathcal{V} = \mathsf{Ab}$ and $\mathcal{V} = \mathsf{Set}$, this is true in the category of all monoids, not just the commutative ones. The case $\mathcal{V} = \mathsf{Ab}$ follows from the following proposition of Silver [56, Prop. 1.1] whose proof is omitted.

Proposition 3.3.8 (Silver)**.** *A morphism $f\colon R \to S$ in $\mathsf{Ring} = \mathsf{Ab}$-$\mathsf{Mon}$ is an epimorphism iff the map $S \otimes_R S \to S$ induced by the multiplication of S is an isomorphism, i.e. iff f is Cauchy dense.*

Corollary 3.3.9. *A morphism $f\colon A \to B$ in Mon is an epimorphism iff it is Cauchy dense.*

Proof. The backwards implication is part of Lemma 3.3.5, so we prove the forwards one. The free abelian group functor $F\colon \mathsf{Set} \to \mathsf{Ab}$ is a strong monoidal left adjoint. Moreover, it reflects isomorphisms by Proposition 2.4.11. Being additionally cocontinuous, it reflects all colimits which exist in Set, i.e. all small colimits. On categories of monoids, F induces the free ring functor $F_*\colon \mathsf{Mon} \to \mathsf{Ring}$, which is again a left adjoint. If $f\colon A \to B$ in Mon is an epimorphism, then so is F_*f. By Proposition 3.3.8, F_*f is Cauchy dense, and then so is f by Proposition 3.1.12. ☐

Silver's result and its corollary give a representation theoretic understanding of epimorphisms of rings and monoids. Indeed, by Theorem 3.1.8, a map of rings $f\colon R \to S$ is an epimorphism iff the restriction of scalars functor $f^*\colon S\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}$ is fully faithful, and similarly for monoid maps.

The coincidence of the epimorphisms in $\mathcal{V}$-Mon and the Cauchy dense functors does not hold for a general monoidal $\mathcal{V}$ with reflexive coequalisers preserved by the product in each variable. We will provide a counterexample taking $\mathcal{V}$ to be the category $\mathcal{F}$ of finitary endofunctors of Set, with its monoidal structure given by composition. Monoids in $\mathcal{V}$ are then finitary monads on Set.

Before giving the counterexample, however, we give a more general characterisation of Cauchy dense maps of monads. Let $\mathcal{A}$ be a cocomplete ordinary category. Then $[\mathcal{A}, \mathcal{A}]$ is cocomplete and strict monoidal under composition. Since colimits of functors are computed objectwise, $- \circ F$ preserves them for any $F \in [\mathcal{A}, \mathcal{A}]$. However, $F \circ -$ need not preserve reflexive coequalisers, so that $[\mathcal{A}, \mathcal{A}]$ does not satisfy our base assumptions for $\mathcal{V}$. A way to fix this is passing to the full subcategory $[\mathcal{A}, \mathcal{A}]_{\mathrm{rc}}$ of endofunctors which preserve reflexive coequalisers. Then $[\mathcal{A}, \mathcal{A}]_{\mathrm{rc}}$-$\mathsf{Mon}$ is the category of reflexive-coequaliser-preserving monads on $\mathcal{A}$.

Remark 3.3.10. If $\mathcal{A}$ is large, the category $[\mathcal{A}, \mathcal{A}]_{\mathrm{rc}}$ may not be locally small. In practice, we will take an even smaller subcategory, such as the category of finitary endofunctors of Set (see Proposition 3.3.12).

Theorem 3.3.11. *Let $\theta\colon S \to T$ be a map of monads on a cocomplete ordinary category $\mathcal{A}$ such that T preserves reflexive coequalisers. Let $U^\theta\colon \mathcal{A}^T \to \mathcal{A}^S$ be the corresponding algebraic functor. Then θ is Cauchy dense iff U^θ is fully faithful.*

Proof. By Theorem 2.4.4, U^θ has a left adjoint F^θ. Its value at $(A, \alpha) \in \mathcal{A}^S$ is given by the reflexive coequaliser in $\mathcal{A}^T$

$$\begin{tikzcd}
(TSA, \mu^T_{SA}) \arrow[r, "T\theta_A"] \arrow[rr, bend right, "T\alpha"'] & (T^2A, \mu^T_{TA}) \arrow[r, "\mu^T_A"] & (TA, \mu^T_A) \arrow[r, two heads] & F^\theta(A, \alpha).
\end{tikzcd} \tag{3.3}$$

Since $\mathcal{A}$ has and T preserve reflexive coequalisers, reflexive coequalisers in $\mathcal{A}^T$ are created (and hence preserved) by U^T. In particular, applying U^T to (3.3), it remains a coequaliser diagram. Taking (A, α) to be $U^\theta F^T A = (TA, \mu^T_A \cdot \theta_{TA}) \in \mathcal{A}^S$, we see that $U^T F^\theta U^\theta F^T A$ is the coequaliser in $\mathcal{A}$ of the pair

$$\begin{tikzcd}
TSTA \arrow[r, "T\theta_{TA}"] & T^3A \arrow[r, shift left, "\mu^T_{TA}"] \arrow[r, shift right, "T\mu^T_A"'] & T^2A.
\end{tikzcd}$$

This is precisely the value of $T \otimes_S T$ at A. Since colimits of endofunctors of $\mathcal{A}$ are computed pointwise, this shows that $T \otimes_S T \cong U^T F^\theta U^\theta F^T$. One checks that the natural transformation $T \otimes_S T \to T$ induced by μ^T becomes $U^T \epsilon F^T$ under this identification, where ϵ is the counit of $F^\theta \dashv U^\theta$.

By definition, θ is Cauchy dense iff $U^T \epsilon F^T$ is an isomorphism. Since U^T reflects isomorphisms, this is equivalent to ϵF^T being an isomorphism. This means that for each $A \in \mathcal{A}$ and $B \in \mathcal{A}^T$, the functor U^θ gives a bijection

$$\mathcal{A}^T(F^T A, B) \cong \mathcal{A}^S(U^\theta F^T A, U^\theta B). \tag{3.4}$$

In fact, this is further equivalent to U^θ being fully faithful. Every $(A, \alpha) \in \mathcal{A}^T$ is the coequaliser of a U^T-split pair $F^T T A \rightrightarrows F^T A$, which we denote as a diagram in $\mathcal{A}^T$ by P_α. Then

$$\begin{aligned}
\mathcal{A}^T((A, \alpha), B) &\cong \mathcal{A}^T(\operatorname{colim} P_\alpha, B) \\
&\cong \lim \mathcal{A}^T(P_\alpha, B) \\
&\cong \lim \mathcal{A}^S(U^\theta P_\alpha, U^\theta B) \\
&\cong \mathcal{A}^S(\operatorname{colim} U^\theta P_\alpha, U^\theta B) \\
&\cong \mathcal{A}^S(U^\theta \operatorname{colim} P_\alpha, U^\theta B) \\
&\cong \mathcal{A}^S(U^\theta(A, \alpha), U^\theta B),
\end{aligned}$$

where the third isomorphism follows from (3.4) and the fact that the objects in the image of P_α are free T-algebras, and the fifth is the fact that U^θ preserves U^T-split coequalisers by Proposition 2.4.3. Of course, if U^θ is fully faithful, then (3.4) holds. □

The maps of monads $\theta\colon S \to T$ such that U^θ is fully faithful were called **dense** by Karazeris and Velebil [32]. The previous theorem then shows that this is equivalent to θ being Cauchy dense, at least when restricting to the reflexive-coequaliser-preserving monads.

The prime context in which Theorem 3.3.11 applies is when taking $\mathcal{V}$ to be the category of sifted-colimit-preserving endofunctors of a locally strongly finitely presentable category under composition – see Lack and Rosický's treatment in [40, §4]. A locally strongly finitely presentable category is, up to equivalence, the free cocompletion of a small category under sifted colimits. This ensures that such a $\mathcal{V}$ is locally small, cocomplete and has a monoidal product which preserves sifted-colimits (and in particular reflexive coequalisers) in each variable.

To find an epimorphism of monads which is not Cauchy dense, we will take $\mathcal{V}$ to be $\mathcal{F}$, the category of finitary endofunctors of Set. Since the inclusion $J\colon \mathsf{FinSet} \to \mathsf{Set}$ is a free cocompletion under filtered colimits, $\mathcal{F}$ is equivalent to $[\mathsf{FinSet}, \mathsf{Set}]$ (through restriction and left Kan extension) and hence it is a complete and cocomplete locally small category. Since colimits commute with colimits, a colimit in $[\mathsf{Set}, \mathsf{Set}]$ of finitary endofunctors is finitary, so it is reflected to a colimit in $\mathcal{F}$. This shows that colimits in $\mathcal{F}$ are computed as in $[\mathsf{Set}, \mathsf{Set}]$, i.e. objectwise. Monoids in $\mathcal{F}$ are, by definition, finitary monads on Set.

To use Theorem 3.3.11, and for $\mathcal{F}$ to satisfy our base assumptions for $\mathcal{V}$, we need the following result. The strategy for the proof is due to Kelly and Schmitt, as they explain after Observation 8.13 in [35].

Proposition 3.3.12. *The inclusion $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$ is a free cocompletion under sifted colimits. In particular, finitary endofunctors of* Set *coincide with those preserving sifted colimits, and hence preserve reflexive coequalisers.*

Proof. The doctrine of finite products is sound, as stated by Adámek, Borceux, Lack and Rosický in [1, Ex. 2.3(ii)]. By their Proposition 3.3, the free cocompletion of a small category $\mathcal{A}$ under sifted colimits is the Yoneda embedding of $\mathcal{A}$ into the category of sifted-flat presheaves $\mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$. By their Theorem 2.4, if $\mathcal{A}^{\mathrm{op}}$ has finite products, then this is just the category of finite-product-preserving functors $\mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$, denoted by $[\mathcal{A}^{\mathrm{op}}, \mathsf{Set}]_{\mathrm{fp}}$.

Now, $\mathsf{FinSet}^{\mathrm{op}}$ has finite limits, so its free cocompletion under sifted colimits is $y\colon \mathsf{FinSet} \hookrightarrow [\mathsf{FinSet}^{\mathrm{op}}, \mathsf{Set}]_{\mathrm{fp}}$. But FinSet is the free cocompletion of the terminal category 1 under finite coproducts, so

$$[\mathsf{FinSet}^{\mathrm{op}}, \mathsf{Set}]_{\mathrm{fp}} \simeq [1, \mathsf{Set}] = \mathsf{Set}.$$

The equivalence is given by evaluating at the singleton set, so that y becomes J, as required.

Since J is also a free cocompletion under filtered-colimits, every finitary endofunctor F of Set is isomorphic to $\mathrm{Lan}_J F'$ for some $F'\colon \mathsf{FinSet} \to \mathsf{Set}$. By the first result, $\mathrm{Lan}_J F'$ preserves sifted colimits. Clearly, every sifted-colimit-preserving functor is finitary. □

Lastly, we need the following characterisation of epimorphisms of finitary monads on Set. It is a special case of Karazeris and Velebil's [32, Prop. 5.1].

Proposition 3.3.13 (Karazeris and Velebil)**.** *A morphism $\theta\colon S \to T$ of finitary monads on* Set *is an epimorphism iff U^θ is injective on objects.*

We are now ready for our counterexample, which simply builds Karazeris and Velebil's Example 5.3.

Example 3.3.14. The forgetful functor from the category of monoids to the category of semigroups is the algebraic functor corresponding to a monad morphism $\theta \colon S \to T$, where S and T are the free semigroup and free monoid monad, respectively.

A semigroup may have at most one identity, and so at most one monoid structure. Thus, U^θ is injective on objects and, by the previous proposition, θ is an epimorphism.

However, U^θ is not fully faithful. For example, let $X = \{0, 1\}$ seen as a multiplicative submonoid of $\mathbb{N}$. The function $X \to X$ which is constant at 0 is a map of semigroups, but not of monoids. By Theorem 3.3.11, θ is not Cauchy dense.

3.3.4 Connected components

This subsection focuses on the $\mathcal{V} = \mathsf{Set}$ case. In particular, we will show that a Cauchy dense ordinary functor induces a bijection between the sets of connected components. In this subsection, we will write $\pi_0 \colon \mathsf{Cat} \to \mathsf{Set}$ for the connected components functor.

Theorem 3.3.15. *Let $F \colon \mathcal{A} \to \mathcal{B}$ be a Cauchy dense ordinary functor. Then $\pi_0 F$ is bijective.*

Proof. First, assume F is bijective-on-objects, and write F^{-1} for the inverse of its object function. Let $b, b' \in \mathcal{B}$. The coend $(F_* \otimes F^*)(b, b')$ can be written as a disjoint union of coends, one for each connected component of $\mathcal{A}$:

$$(F_* \otimes F^*)(b, b') = \coprod_{C \in \pi_0 \mathcal{A}} \int^{a \in C} \mathcal{B}(Fa, b') \times \mathcal{B}(b, Fa). \tag{3.5}$$

A simple way to see this is that the functor that sends a category $\mathcal{C}$ to the opposite of its subdivision category, in the terminology of Mac Lane [48, p. 224], preserves coproducts.

We will show that a and a' are connected in $\mathcal{A}$ iff Fa and Fa' are connected in $\mathcal{B}$. The forward implication is true for any functor. Now suppose there is a morphism $f \colon Fa \to Fa'$ in $\mathcal{B}$. We have two equivalence classes $[f, 1_{Fa}]$ and $[1_{Fa'}, f]$ in $(F_* \otimes F^*)(Fa, Fa')$ which are mapped to f by $\epsilon_{Fa,Fa'}$. Since F is Cauchy dense, they must actually be the same equivalence class. In particular, $[f, 1_{Fa}]$ and $[1_{Fa'}, f]$ must lie in the same component of the disjoint union in (3.5), i.e. a and a' are connected. If instead of a morphism $Fa \to Fa'$ there is a zigzag of morphisms, one can apply the same argument to each morphism in the zigzag to conclude that a and a' are connected. Since F is bijective-on-objects, it follows that $\pi_0 F$ is bijective.

Now we relax the assumption that F is bijective-on-objects. Write $\mathcal{A} \xrightarrow{G} \operatorname{im} F \xrightarrow{H} \mathcal{B}$ for the factorisation of F into a bijective-on-objects functor followed by a fully faithful one. By Proposition 3.1.10(b), since H is fully faithful, G is Cauchy dense, and then H is Cauchy dense by part (c) of the same proposition. We have already shown that $\pi_0 G$ is bijective. By Theorem 3.2.5, $\mathcal{B}$ is a full subcategory of the idempotent completion $\overline{\operatorname{im} F}$ of $\operatorname{im} F$ containing $\operatorname{im} F$ itself. In other words, the objects of $\mathcal{B}$ include all of $\operatorname{im} F$ and the splitting of some (but possibly not all) idempotents in $\operatorname{im} F$. Splitting certain idempotents does not change the connected components of $\operatorname{im} F$, so we conclude that $\pi_0 H$ is bijective. $\square$

Note that the conclusion of the previous theorem fails if we replace the Cauchy dense functor F by a merely (co)dense one, even if we ask that it be fully faithful. A simple example is given by freely adjoining the (co)product of the two objects in the discrete category 2.

Corollary 3.3.16. *An ordinary Cauchy dense functor $F\colon \mathcal{A} \to \mathcal{B}$ can be written as $\bigsqcup_i F_i\colon \bigsqcup_i \mathcal{A}_i \to \bigsqcup_i \mathcal{B}_i$ where each F_i is Cauchy dense and each $\mathcal{A}_i$ and $\mathcal{B}_i$ are connected. Conversely, a functor of the form $\bigsqcup_i G_i$ is Cauchy dense iff each G_i is Cauchy dense.*

Proof. Since $\pi_0 F$ is a bijection, $F = \bigsqcup_{i \in \pi_0 \mathcal{A}} F_i$ where F_i is the restriction of F to $i \in \pi_0 \mathcal{A}$ factorised through $\pi_0 F(i) \in \pi_0 \mathcal{B}$. It is easy to see that each F_i is Cauchy dense if F is from (3.5). Similarly, it is clear that a disjoint union of Cauchy dense functors is Cauchy dense. $\square$

Proposition 3.3.17. *Let $f\colon A \to B$ be a monoid homomorphism with A a group. Then f is Cauchy dense iff it is surjective. If this is the case, then B is also a group.*

Proof. By Lemma 3.3.5, f is Cauchy dense iff, for all $b \in B$, $(b, 1)$ and $(1, b)$ in $B \times B$ become related under the quotient map to $B \otimes_A B$. This is the quotient by the equivalence relation generated by setting $(bf(a), b') \sim (b, f(a)b')$ for all $a \in A$ and $b, b' \in B$.

Since A has an identity element, $\sim$ is reflexive. It is also transitive, because we can compose in A. Lastly, since A is a group we have

$$(b, f(a)b') = (bf(a)f(a^{-1}), f(a)b') \sim (bf(a), f(a^{-1})f(a)b') = (bf(a), b'),$$

so $\sim$ is also symmetric. Thus, $\sim$ is already an equivalence relation.

Hence, f is Cauchy dense iff for all $b \in B$ we have $(b, 1) \sim (1, b)$, iff for all $b \in B$ there is some $a \in A$ such that $b = f(a)$. $\square$

Corollary 3.3.18. *A functor whose domain is a groupoid is Cauchy dense iff it is equivalent to a disjoint union of surjective group homomorphisms.*

Proof. Corollary 3.3.16 reduces our task to showing that a Cauchy dense functor whose domain is a group must be equivalent to a surjective group homomorphism.

Let $F\colon A \to \mathcal{B}$ be Cauchy dense, with A a group, and let $A \xrightarrow{G} \operatorname{im} F \xrightarrow{H} \mathcal{B}$ be its factorisation into a bijective-on-objects functor followed by a fully faithful one. As before, both G and H are Cauchy dense because F is. By Proposition 3.3.17, G is a surjective group homomorphism. By Theorem 3.2.5, $\mathcal{B}$ is equivalent to a full subcategory of the idempotent completion of $\operatorname{im} F$ containing $\operatorname{im} F$. Since $\operatorname{im} F$ is a group, it is already idempotent complete, so H must be an equivalence. Thus, F is equivalent to a surjective group homomorphism. $\square$

Corollary 3.3.19. *A Cauchy dense functor whose domain is discrete is an equivalence.*

3.3.5 Split-full Cauchy dense functors

We conclude by showing that for split-full functors, much like when $\mathcal{V}_0$ is a preorder, Cauchy density is equivalent to an apparently much weaker condition. This simplifies checking the hypotheses of Lemma 3.2.3. The proof, which is straightforward when $\mathcal{V} = \mathsf{Set}$, is slightly more involved in the general case. Note that in the simpler condition below we quantify over a single $b \in \mathcal{B}$, instead of a pair of them, and we ask for a split epimorphism, rather than an isomorphism.

Theorem 3.3.20. *Let $\mathcal{A}$ be a small $\mathcal{V}$-category, and $F \colon \mathcal{A} \to \mathcal{B}$ be a split-full functor. Then F is Cauchy dense iff*

$$\epsilon_{b,b} \colon \int^a \mathcal{B}(Fa, b) \otimes \mathcal{B}(b, Fa) \longrightarrow \mathcal{B}(b, b)$$

is a split epimorphism for all $b \in \mathcal{B}$.

Proof. The forward implication is obvious, so we prove converse.

For $b_0, \dots, b_n \in \mathcal{B}$, we will use the abbreviation

$$\mathcal{B}(b_0, \dots, b_n) := \mathcal{B}(b_{n-1}, b_n) \otimes \cdots \otimes \mathcal{B}(b_0, b_1).$$

We write $M_{b_1} \colon \mathcal{B}(b_0, b_1, b_2) \to \mathcal{B}(b_0, b_2)$ for the composition map, and similarly for longer tuples.

Let $b, b' \in \mathcal{B}$. First, we will show that $\epsilon_{b,b'}$ is a split epimorphism. The square

$$\begin{tikzcd}
\int^a \mathcal{B}(b, Fa, b, b') \arrow[r, "1 \otimes \epsilon_{b,b}"] \arrow[d, "\int^a M_b"'] & \mathcal{B}(b, b, b') \arrow[d, "M_b"] \\
\int^a \mathcal{B}(b, Fa, b') \arrow[r, "\epsilon_{b,b'}"'] & \mathcal{B}(b, b')
\end{tikzcd}$$

commutes by associativity in $\mathcal{B}$. Now, $1 \otimes \epsilon_{b,b}$ is a split epimorphism by assumption, and so is M_b by the right unit axiom in $\mathcal{B}$. It follows that $\epsilon_{b,b'}$ is also a split epimorphism.

It now suffices to show that $\epsilon_{b,b'}$ is monic. The diagram

$$\begin{tikzcd}
\mathcal{B}(Fa, b') \otimes \mathcal{A}(a', a) \otimes \mathcal{B}(b, Fa') \arrow[dr, "1 \otimes F_{a',a} \otimes 1"] & & \\
& \mathcal{B}(b, Fa', Fa, b') \arrow[r, "M_{Fa}"] \arrow[d, "M_{Fa'}"'] & \mathcal{B}(b, Fa', b') \arrow[d, "c_{a'}"] \\
& \mathcal{B}(b, Fa, b') \arrow[r, "c_a"'] & \int^a \mathcal{B}(b, Fa, b')
\end{tikzcd}$$

commutes when c is a cowedge, and in particular when c is the coend cowedge. Since $1 \otimes F_{a',a} \otimes 1$ has a right inverse by assumption, the square alone also commutes. Seeing each of the composites in the square as an appropriate cowedge shows that the two maps

$$\begin{tikzcd}
 & \int^{a} \mathcal{B}(b, Fa, b') \arrow[d, equal] \\
\int^{a,a'} \mathcal{B}(b, Fa', Fa, b') \arrow[ur, "\int^{a} 1 \otimes \epsilon_{b,Fa}"] \arrow[dr, "\int^{a'} \epsilon_{Fa',b'} \otimes 1"'] & \\
 & \int^{a'} \mathcal{B}(b, Fa', b')
\end{tikzcd}$$

coincide.

The naturality of ϵ ensures that the two squares in the diagram

$$\begin{tikzcd}
 & \int^{a,a'} \mathcal{B}(b, Fa', b, Fa, b') \arrow[dl, "\int^{a} 1 \otimes \epsilon_{b,b}"'] \arrow[dd, "\int^{a,a'} M_b"] \arrow[dr, "\int^{a'} \epsilon_{b,b'} \otimes 1"] & \\
\int^{a} \mathcal{B}(b, b, Fa, b') \arrow[dd, "\int^{a} M_b"'] & & \int^{a'} \mathcal{B}(b, Fa', b, b') \arrow[dd, "\int^{a'} M_b"] \\
 & \int^{a,a'} \mathcal{B}(b, Fa', Fa, b') \arrow[dl, "\int^{a} 1 \otimes \epsilon_{b,Fa}"] \arrow[dr, "\int^{a'} \epsilon_{Fa',b'} \otimes 1"'] & \\
\int^{a} \mathcal{B}(b, Fa, b') \arrow[rr, equal] & & \int^{a'} \mathcal{B}(b, Fa', b')
\end{tikzcd}$$

commute, so that the whole diagram commutes. Distributing tensors over coends, the top leftmost map may in fact be written as $1 \otimes \epsilon_{b,b}$, and the top rightmost map is just $\epsilon_{b,b'} \otimes 1$. Both maps on the left-hand side have right inverses. Writing $\epsilon^{r}_{b,b}$ for a right inverse to $\epsilon_{b,b}$, the composite

$$\begin{tikzcd}
\int^{a} \mathcal{B}(b, Fa, b') \arrow[r, "r^{-1}"] & \int^{a} \mathcal{B}(b, Fa, b') \otimes I \arrow[r, "1 \otimes j_b"] & \int^{a} \mathcal{B}(b, b, Fa, b') \arrow[d, "1 \otimes \epsilon^{r}_{b,b}"] \\
\int^{a'} \mathcal{B}(b, Fa', b') & \int^{a'} \mathcal{B}(b, Fa', b, b') \arrow[l, "\int^{a'} M_b"'] & \int^{a,a'} \mathcal{B}(b, Fa', b, Fa, b') \arrow[l, "\epsilon_{b,b'} \otimes 1"']
\end{tikzcd}$$

is the identity, where r is the right unitor of $\mathcal{V}$. Then, for any morphism $x \colon X \to \int^{a} \mathcal{B}(b, Fa, b')$, we have

$$\begin{aligned}
x &= \left(\int^{a'} M_b\right) \cdot (\epsilon_{b,b'} \otimes \epsilon^{r}_{b,b} j_b) \cdot r^{-1} \cdot x \\
&= \left(\int^{a'} M_b\right) \cdot (\epsilon_{b,b'} \otimes \epsilon^{r}_{b,b} j_b) \cdot (x \otimes 1) \cdot r^{-1} \\
&= \left(\int^{a'} M_b\right) \cdot (\epsilon_{b,b'} x \otimes \epsilon^{r}_{b,b} j_b) \cdot r^{-1}.
\end{aligned}$$

In particular, x is determined by $\epsilon_{b,b'} x$, so $\epsilon_{b,b'}$ is monic. □

Of course, a split-full Cauchy dense functor may not be fully faithful. For example, a surjective group homomorphism is split-full and Cauchy dense (by Corollary 3.3.18), but only fully faithful if it is an isomorphism.

Chapter 4

Pushforward monads

The question of how a monad T on a category $\mathcal{A}$ can be transported along a functor $G\colon \mathcal{A} \to \mathcal{B}$ to produce a monad on $\mathcal{B}$ has a well-known answer when G has a left adjoint F. If so, GTF has a monad structure induced from that of T and the adjunction. There is a lesser-known, more general answer however: even if G is not a right adjoint, the right Kan extension of GT along T, when it exists, has a monad structure. The monad obtained thus is the pushforward of T along G, which we denote by $G_\sharp T$. This notion of pushforward monad can be traced back to Street's seminal paper [58, p. 155], but has received little attention since. In this chapter, we explore the general properties of this construction, both in the setting of monads in a 2-category and in that of monads in CAT, the 2-category of locally small categories.

Pushforward monads generalise the concept of codensity monads, which correspond to the case where T is the identity monad on $\mathcal{A}$. This case has been extensively studied, with the most famous example being the codensity monad of the inclusion $\mathsf{FinSet} \hookrightarrow \mathsf{Set}$. Kennison and Gildenhuys [37, p. 341] showed that this is the ultrafilter monad, which had been identified by Manes [50, Prop. 5.5] as the monad for compact Hausdorff spaces. This is a remarkable fact: the theory of compact Hausdorff spaces is obtained from the concept of finite set through some general categorical machinery, without any mention of topology. An account of this and related results was given by Leinster [45]. Other well-known codensity monads of functors without a left adjoint include:

- The codensity monad of the inclusion of finite groups into groups is the profinite completion monad, whose algebras are profinite groups, or equivalently, totally disconnected compact Hausdorff topological groups.

- The codensity monad of the inclusion of finite rings into rings is again the profinite completion monad, whose algebras are profinite rings. These are all the compact Hausdorff topological rings, which are automatically totally disconnected. This and the previous result can be found in Devlin's PhD thesis [19, Thms. 10.1.30 and 10.2.2].

- The Giry monad, which sends a measurable space to the space of all probability measures on it, is the codensity monad of the inclusion of a certain (not full) subcategory of the category of measurable spaces. This and a number of similar results applying to other probability monads can be found in work of Avery [7, Thm. 5.8] and Van Belle [62].

- If T is a finitary monad on Set, then the codensity monad of the inclusion $\mathsf{FinFam}((\mathsf{Set}^T)^{\mathrm{op}}) \hookrightarrow \mathsf{Fam}((\mathsf{Set}^T)^{\mathrm{op}})$ of the free finite coproduct cocompletion of $(\mathsf{Set}^T)^{\mathrm{op}}$ into its free coproduct cocompletion gives the ultraproduct monad of model theory, whose algebras are sheaves of T-algebras on compact Hausdorff spaces (see Section 6.2).

All of these are examples of straightforward functors whose codensity monad give interesting, nontrivial theories. Pushforward monads add a new axis to this list, so that one can now vary both the functor and the monad on the domain being pushed forward.

In Section 4.1, we develop the theory of pushforward monads in a general 2-category, starting from their construction. It includes a universal property satisfied by pushforwards with respect to lax monad functors (Theorem 4.1.5), which was first stated by Street [58, Thm. 5]. This allows us not only to see codensity monads as special cases of pushforwards, but also vice versa.

Section 4.2 identifies a new kind of universal property of pushforwards with respect to colax monad functors (Corollary 4.2.6), which later allows us to generalise some ideas already present in Adámek and Sousa's work on D-ultrafilters [3] (Example 4.3.15(ii)).

In Section 4.3, we specialise these results to the 2-category CAT. We give sufficient conditions for the existence of pushforwards (Proposition 4.3.5) and for the possibility of their iteration (Proposition 4.3.6). One remarkable consequence of the results in the previous two sections is the existence of two adjunctions between categories of monads, the most important one being the content of Theorem 4.3.11. A particular instantiation of this adjunction shows that the concept of the pseudoconstants of a theory appears naturally from the consideration of pushforwards along $\mathsf{Set}_{\geq 1} \hookrightarrow \mathsf{Set}$ (Example 4.3.14). The last subsection shows that codensity monads are stable in a suitable sense under limit completions (Proposition 4.3.17), and relates them to Diers's theory of multiadjunctions [21, 23], and Tholen's $\mathfrak{C}$-pro-adjunctions [61]. This allows us, for example, to identify the category of algebras of the codensity monad of the inclusion $\mathsf{Field} \hookrightarrow \mathsf{CRing}$ as the free product completion of the category of fields (Example 4.3.23(i)).

4.1 Pushforward monads

In this section, we spell out the construction of the pushforward of a monad and give its general properties. Although it is largely expository, some of the results here, such as Lemmas 4.1.4 and 4.1.7, do not seem to have appeared in print.

We work in the setting of a 2-category $\mathcal{K}$, although from Section 4.3 onwards we will take $\mathcal{K}$ to be CAT. The pushforward construction makes sense in any bicategory, but we will do everything in the strict case for simplicity without any loss of generality. We use the same notation as in Section 2.1, i.e. $\alpha \cdot \beta$ for the vertical composite of two 2-cells α and β, and $\alpha\beta$ for their horizontal composite.

Given a 0-cell z and a 1-cell $g\colon x \to y$ in $\mathcal{K}$, recall that we write $g_\circ\colon \mathcal{K}(z,x) \to \mathcal{K}(z,y)$ and $g^\circ\colon \mathcal{K}(y,z) \to \mathcal{K}(x,z)$ for the functors given by composing with g on the left and on the right, respectively. Given a monad t on x, and taking $z = y$, we can form the comma category $g^\circ{\downarrow}gt$ whose objects are pairs (s,σ) such that $s \in \mathcal{K}(y,y)$ and $\sigma\colon sg \to gt$.

Proposition 4.1.1. *Let t be a monad on x, and $g\colon x \to y$ be a 1-cell. The comma category $g^\circ{\downarrow}gt$ admits a strict monoidal structure such that the forgetful functor $g^\circ{\downarrow}gt \to \mathcal{K}(y,y)$ is strict monoidal.*

Proof. Given two objects (s,σ) and (s',σ') of $g^\circ{\downarrow}gt$, we define $(s',\sigma') \otimes (s,\sigma)$ to be the composite

$$\begin{tikzcd}
x \arrow[r, "g"] \arrow[d, "t"] \arrow[dd, bend right=60, "t"'] & y \arrow[d, "s"] \arrow[dl, Rightarrow, "\sigma"'] \\
\overset{\mu^t}{\Leftarrow}\; x \arrow[r, "g"] \arrow[d, "t"] & y \arrow[d, "s'"] \arrow[dl, Rightarrow, "\sigma'"'] \\
x \arrow[r, "g"] & y.
\end{tikzcd}$$

The monoidal unit is simply $g\eta^t\colon g \to gt$. A routine calculation shows that the axioms for a strict monoidal category follow from the axioms for the monad t. That the forgetful functor $g^\circ{\downarrow}gt \to \mathcal{K}(y,y)$ is strict monoidal is clear. $\square$

A right extension of gt along g is precisely a terminal object of $g^\circ{\downarrow}gt$. As the terminal object of a monoidal category, it has a unique monoid structure. Taking its image under the forgetful functor $g^\circ{\downarrow}gt \to \mathcal{K}(y,y)$ gives a monoid in $\mathcal{K}(y,y)$, i.e. a monad on y.

Definition 4.1.2. Let t be a monad on x and $g\colon x \to y$ be a 1-cell. The **pushforward** of t along g is the right extension of gt along g, with its canonical monad structure described above. When it exists, we denote it by $g_\sharp t$.

We can give a more explicit description of the monad structure of $g_\sharp t$. Let $(g_\sharp t, \epsilon)$ be a right extension of gt along g, as depicted by the diagram

$$\begin{tikzcd}
x \arrow[r, "g"] \arrow[d, "t"'] & y \arrow[d, "g_\sharp t"] \arrow[dl, Rightarrow, "\epsilon"'] \\
x \arrow[r, "g"'] & y.
\end{tikzcd}$$

We call ϵ the **counit** of the pushforward $g_\sharp t$. The unit and multiplication of $g_\sharp t$ are the unique 2-cells $\eta^{g_\sharp t}$ and $\mu^{g_\sharp t}$ making

$$\begin{tikzcd}
& g \arrow[dl, "(\eta^{g_\sharp t})g"'] \arrow[dr, "g\eta^t"] & \\
(g_\sharp t)g \arrow[rr, "\epsilon"'] & & gt
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
(g_\sharp t)^2 g \arrow[r, "(g_\sharp t)\epsilon"] \arrow[d, "\mu^{g_\sharp t}g"'] & (g_\sharp t)gt \arrow[r, "\epsilon t"] & gt^2 \arrow[d, "g\mu^t"] \\
(g_\sharp t)g \arrow[rr, "\epsilon"'] & & gt
\end{tikzcd}
\tag{4.1}$$

commute. Note that the commutativity of these diagrams means that (g, ϵ) is a lax monad functor $(x, t) \to (y, g_\sharp t)$.

Examples 4.1.3.

(i) If $g\colon x \to y$ is right adjoint to f, with unit δ and counit ν, recall from Lemma 2.2.3(iii) that the right extension of a 1-cell $h\colon x \to z$ along g is $(hf, h\nu)$. In particular, $g_\sharp t = gtf$, with its well-known monad structure:

$$\eta^{g_\sharp t} = g\eta^t f \cdot \delta \qquad \text{and} \qquad \mu^{g_\sharp t} = g\mu^t f \cdot gt\nu tf.$$

(ii) The pushforward of the identity monad along g is the **codensity monad** of g. In this case, the underlying endofunctor of the monad is the right extension of g along itself. By the previous example, the codensity monad of a right adjoint is the monad induced by the adjunction.

(iii) Let $\mathbf{1}$ be the terminal category. There is a unique monad on $\mathbf{1}$, namely the identity monad. A functor $G\colon \mathbf{1} \to \mathcal{A}$ corresponds to an object x of $\mathcal{A}$. If $\mathcal{A}$ is locally small and has Set-indexed powers, then the codensity monad of G is given by $a \mapsto x^{\mathcal{A}(a,x)}$. This is the **endomorphism monad** of x, introduced in Example 2.4.15 for $\mathcal{A} = \mathsf{Set}$. Proposition 4.1.5 and the discussion following it show that for any monad T on $\mathcal{A}$, the T-algebra structures on x correspond to monad maps from T to the endomorphism monad of x.

(iv) Let $\mathcal{V}$ be a monoidal category. Monads in the one-object bicategory $\mathrm{B}\mathcal{V}$ are monoids in $\mathcal{V}$, and right extension along an object $x \in \mathcal{V}$ is by definition a right adjoint to $- \otimes x$. If m is a monoid, and $- \otimes x \dashv [x, -]$, then the pushforward $x_\sharp m$ is $[x, x \otimes m]$, with unit and multiplication the respective transposes of

$$\begin{tikzcd} x \arrow[d, "x \otimes \eta^m"] \\ x \otimes m \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} {[x, x \otimes m] \otimes [x, x \otimes m] \otimes x} \arrow[d, "{[x, x \otimes m] \otimes \mathrm{ev}_{x \otimes m}}"] \\ {[x, x \otimes m] \otimes x \otimes m} \arrow[d, "\mathrm{ev}_{x \otimes m} \otimes m"] \\ x \otimes m \otimes m \arrow[d, "x \otimes \mu^m"] \\ x \otimes m. \end{tikzcd}$$

The pushforward construction is functorial in the monad argument. In fact, the induced functor between categories of monads is as continuous as the 1-cell along which one pushes forward, as shown by the next lemma. It is not quite functorial in the 1-cell argument, but it does preserve composition laxly (Lemma 4.1.7). Recall that we write Mnd_x for the category of monads on a 0-cell x of $\mathcal{K}$ and monad maps between them (Definition 2.3.3).

Lemma 4.1.4. *Let $g\colon x \to y$ be a 1-cell. Then $g_\sharp$ is a functor $\mathsf{Mnd}_x \to \mathsf{Mnd}_y$ insofar as it is defined. Moreover, $g_\sharp$ preserves the limit of any diagram $D\colon \mathcal{I} \to \mathsf{Mnd}_x$ such that $g_\circ$ preserves the limit of $\mathcal{I} \xrightarrow{D} \mathsf{Mnd}_x \to \mathcal{K}(x,x)$.*

Proof. Let t and s be monads on x such that $g_\sharp t$ and $g_\sharp s$ exist, with counits ϵ^t and ϵ^s respectively. Given $\theta \in \mathsf{Mnd}_x(t,s)$, there is a unique 2-cell $g_\sharp\theta\colon g_\sharp t \to g_\sharp s$ such that $\epsilon^s \cdot (g_\sharp\theta)g = g\theta \cdot \epsilon^t$. This is a monad map: the equation

$$\epsilon^s \cdot (g_\sharp\theta \cdot \eta^{g_\sharp t})g = g\theta \cdot \epsilon^t \cdot \eta^{g_\sharp t}g \overset{(4.1)}{=} g\theta \cdot g\eta^t \overset{(\theta.\eta)}{=} g\eta^s \overset{(4.1)}{=} \epsilon^s \cdot \eta^{g_\sharp s}g$$

implies that $g_\sharp\theta \cdot \eta^{g_\sharp t} = \eta^{g_\sharp s}$ by uniqueness of factorisations through ϵ^s. The proof that $g_\sharp\theta$ is compatible with the multiplication is similar. As is often the case, the uniqueness of $g_\sharp\theta$ guarantees functoriality.

For the statement about of limits, recall that the functor $U_x\colon \mathsf{Mnd}_x \to \mathcal{K}(x,x)$ creates limits by Lemma 2.3.6. Let $\lambda\colon l \to D$ be a limit cone. Then $g_\circ U_x \lambda\colon g_\circ U_x l \to g_\circ U_x D$ is a limit cone in $\mathcal{K}(x,y)$ by assumption. Since ran_g is a partial right adjoint to g°, it preserves limits whenever it is defined, so $\operatorname{ran}_g g_\circ U_x \lambda$ is still a limit cone. This is precisely the image under $U_y\colon \mathsf{Mnd}_y \to \mathcal{K}(y,y)$ of the cone $g_\sharp\lambda\colon g_\sharp l \to g_\sharp D$, which is then a limit cone since U_y reflects limits. ☐

Pushforward monads were first introduced by Street in [58, p. 155]. In fact, he defines them in terms of a universal property (given in the next theorem) with respect to the forgetful functor $\mathrm{Und}\colon \mathsf{MND}(\mathcal{K}) \to \mathcal{K}$ from the 2-category of monads in $\mathcal{K}$ introduced in Section 2.3. He then shows that the right extension in Definition 4.1.2 has this universal property in Theorem 5 of the same article.

Recall from Definition 2.3.3 that we write $\mathsf{Lax}(t,s)$ for the collection of lax monad functors between monads t and s in $\mathcal{K}$, and $\mathsf{Lax}_g(t,s)$ for the subset of those whose 1-cell part is g. Similarly, we have $\mathsf{Colax}(t,s)$ and $\mathsf{Colax}_g(t,s)$.

Theorem 4.1.5 (Street)**.** *Let $g\colon x \to y$ be a 1-cell, t be a monad on x, and s be a monad on y. If $g_\sharp t$ exists, then there is a bijection*

$$\mathsf{Lax}_g(t,s) \cong \mathsf{Mnd}_y(s, g_\sharp t)$$

natural in t and s.

Proof. Let ϵ be the counit of $g_\sharp t$, so that $(g,\epsilon) \in \mathsf{Lax}(t, g_\sharp t)$. A monad map $\theta\colon s \to g_\sharp t$ is in particular a lax monad functor $(1_y, \theta)\colon g_\sharp t \to s$, so composing with (g,ϵ) gives an assignment $\mathsf{Mnd}_y(s, g_\sharp t) \to \mathsf{Lax}_g(t,s)$. Given $(g,\varphi) \in \mathsf{Lax}_g(t,s)$, the universal property of the right extension defining $g_\sharp t$ gives a unique 2-cell $\widehat{\varphi}\colon s \to g_\sharp t$ such that $\varphi = \epsilon \cdot \widehat{\varphi}g$. It suffices to check that $\widehat{\varphi}$ is a monad map, since then this construction gives an inverse to the former assignment. We have

$$\epsilon \cdot (\widehat{\varphi} \cdot \eta^s)g = \epsilon \cdot \widehat{\varphi}g \cdot \eta^s g = \varphi \cdot \eta^s g \overset{(\varphi.\eta)}{=} g\eta^t \overset{(4.1)}{=} \epsilon \cdot \eta^{g_\sharp t}g,$$

which implies that $\widehat{\varphi} \cdot \eta^s = \eta^{g_\sharp t}$ by uniqueness of factorisations through ϵ. The proof that $\widehat{\varphi}$ is compatible with the multiplication is similar.

Naturality in t is with respect to monad maps in Mnd_x, while that in s is with respect to monad maps in Mnd_y. They both follow easily from the fact that the bijection is given by composition with (g, ϵ) and that $g_\sharp$ is a functor. $\square$

Street stated the previous universal property as the fact that $(g, \epsilon)\colon (x, t) \to (y, g_\sharp t)$ is pre-opcartesian with respect to $\mathrm{Und}\colon \mathsf{MND}(\mathcal{K}) \to \mathcal{K}$.

Remark 4.1.6. Recall that $\mathsf{MND}(\mathcal{K})$ is a 2-category, so both sides of the bijection in the previous theorem are actually categories. In each case, the morphisms can be taken to be the corresponding 2-cells in $\mathsf{MND}(\mathcal{K})$. However, the bijection above does not lift to an isomorphism (or even an equivalence) of categories. There is a functor going left, because that direction is given by composing with a 1-cell in $\mathsf{MND}(\mathcal{K})$, but there is no functor going the other way in general. Such a functor would need to, for example, send an endo-2-cell of g to an endo-2-cell of 1_y.

The significance of this universal property becomes clear when $\mathcal{K}$ admits the construction of algebras. Translating the above bijection along the locally fully faithful 2-functor $\mathsf{MND}(\mathcal{K}) \hookrightarrow \mathcal{K}^{\to}$ of Theorem 2.3.7 gives a correspondence between the collections of dashed 1-cells which make each of the following diagrams in $\mathcal{K}$ commute:

$$\begin{tikzcd} x^t \arrow[r, dashed] \arrow[d, "u^t"'] & y^s \arrow[d, "u^s"] \\ x \arrow[r, "g"'] & y \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} y^{g_\sharp t} \arrow[rr, dashed] \arrow[dr, "u^{g_\sharp t}"'] & & y^s \arrow[dl, "u^s"] \\ & y. & \end{tikzcd} \tag{4.2}$$

In the terminology of Sections 1.2 and 2.3, this means that $u^{g_\sharp t}$ is the monadic reflection of gu^t.

Lemma 4.1.7. *Let $g\colon x \to y$ and $h\colon y \to z$ be 1-cells, and t a monad on x. Then there exists a canonical map of monads on z*

$$h_\sharp(g_\sharp t) \to (hg)_\sharp t$$

natural in t, assuming the domain and codomain exist. Moreover, this map is an isomorphism if h preserves $\mathrm{ran}_g\, gt$.

Proof. Let ϵ^g, ϵ^h and ϵ^{hg} be the counits of $g_\sharp t$, $h_\sharp(g_\sharp t)$ and $(hg)_\sharp t$, respectively. Since $(g, \epsilon^g) \in \mathsf{Lax}(t, g_\sharp t)$ and $(h, \epsilon^h) \in \mathsf{Lax}(g_\sharp t, h_\sharp(g_\sharp t))$, their composite $(hg, \epsilon^h g \cdot h\epsilon^g)$ is a lax monad functor $t \to h_\sharp(g_\sharp t)$ whose 1-cell part is hg. By Theorem 4.1.5, this corresponds to a unique monad map $h_\sharp(g_\sharp t) \to (hg)_\sharp t$. Naturality follows from the fact that pushforward is a functor, and that the bijection in Theorem 4.1.5 is natural in t.

If h preserves $\mathrm{ran}_g\, gt$, the right extension of hgt along g is given by $(h(\mathrm{ran}_g\, gt), h\epsilon^g)$. By Lemma 2.2.3(iv), $\mathrm{ran}_h(\mathrm{ran}_g\, hgt) \cong \mathrm{ran}_{hg}\, hgt$ and the former may now be given as $(\mathrm{ran}_h\, h(\mathrm{ran}_g\, gt), \epsilon^h g \cdot h\epsilon^g)$. This is to say that the factorisation of $\epsilon^h g \cdot h\epsilon^g$ through ϵ^{hg} is an isomorphism, as needed. $\square$

Informally, this lemma (together with the identification $t = x_\sharp t$) can be interpreted as saying that the assignment $x \mapsto \mathsf{Mnd}_x$ and $g \mapsto g_\sharp$ is a normal[1] lax

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[1] In the sense that it preserves identity 1-cells strictly.

functor from $\mathcal{K}_0$ (the underlying 1-category of $\mathcal{K}$) to a 2-category of categories, (partial) functors and natural transformations.

Remark 4.1.8. In the notation of Lemma 4.1.7, for h to preserve $\mathrm{ran}_g\, gt$ it suffices for one of g and h to be a right adjoint by Lemma 2.2.3. In CAT, it is also sufficient for $\mathrm{ran}_g\, gt$ to be pointwise, and for h to preserve the limits involved. This happens, for example, if x is a small category, y is complete and h preserves all small limits.

For an example where the map is not an isomorphism, one can take $\mathcal{K} = \mathsf{CAT}$, g to be the unique functor $\mathsf{0} \to \mathsf{1}$ from the initial to the terminal category, h to be the functor $\mathsf{1} \to \mathsf{Set}$ picking out a set X with at least two elements, and t to be the identity monad on $\mathsf{0}$. Then $g_\sharp t$ is the identity monad on $\mathsf{1}$, and $h_\sharp(g_\sharp t)$ is the endomorphism monad of X (Example 4.1.3(iii)) while $(hg)_\sharp t$ is constant at the terminal set.

Note that (4.2) says that $u^{g_\sharp t}$ is the monadic reflection of gu^t, but the same is true for $u^{(gu^t)_\sharp 1}$ assuming $(gu^t)_\sharp 1$ exists. Since u^t is a right adjoint, the previous remark and lemma imply that $(gu^t)_\sharp 1 \cong g_\sharp(u^t_\sharp 1)$, and by Example 4.1.3(i) we have $u^t_\sharp 1 = t$. Altogether, this shows:

Corollary 4.1.9. *Let $\mathcal{K}$ admit the construction of algebras, $g\colon x \to y$ be a 1-cell in $\mathcal{K}$, and t a monad on x. Then $g_\sharp t$ is the codensity monad of gu^t.*

This corollary seems to suggest that the study of pushforward monads is subsumed by that of codensity monads. However, the pushforward construction has several benefits that are lost by passing to the associated codensity monad. Most importantly, pushforwards give a functor between categories of monads, and so can be used to produce maps between complex monads from maps between simple ones.

4.2 Colax monad functors and g-determined monads

In this section, we study a different universal property that pushforwards enjoy with respect to colax monad functors. It was inspired by Adámek and Sousa's work in [3], where they give explicit descriptions of the codensity monads of several functors of the form $\mathcal{A}_{\mathrm{fp}} \hookrightarrow \mathcal{A}$, where $\mathcal{A}$ is a locally finitely presentable category and $\mathcal{A}_{\mathrm{fp}}$ is its full subcategory on the finitely presentable objects. They say a monad T on $\mathcal{A}$ has the *limit property* with respect to a full embedding $J\colon \mathcal{B} \hookrightarrow \mathcal{A}$ if $(T, 1_{TJ})$ is the pointwise right Kan extension of TJ along J. They then exhibit a certain monad S with this property, and show that the codensity monad of J is the smallest submonad of S with the limit property. Our work in this section will generalise this to the context of pushforward monads in a 2-category. For a more explicit comparison, see Example 4.3.15(ii) below.

Definition 4.2.1. Let $g\colon x \to y$ and $h\colon y \to z$ be 1-cells. We say that h is **g-determined** if $(h, 1_{hg})$ is a right extension of hg along g. Dually, g is **h-opdetermined** if it is h-determined in $\mathcal{K}^{\mathrm{op}}$, i.e. if $(g, 1_{hg}) = \mathrm{rift}_h\, hg$. A monad on y is g-(op)determined if its 1-cell part is.

Example 4.2.2. Let $g\colon x \to y$ be right adjoint to f, with unit η and counit ϵ. Then $h\colon y \to z$ is g-determined iff $h\eta$ is an isomorphism. Indeed, we have $\operatorname{ran}_g hg = (hgf, hg\epsilon)$ by Lemma 2.2.3(iii). The claim follows easily from the fact that $h\eta$ is the unique morphism $(h, 1_{hg}) \to (hgf, hg\epsilon)$ in $g^\circ{\downarrow}hg$, since the codomain is terminal.

It follows that f is g-determined iff the adjunction $f \dashv g$ is idempotent. Moreover, if f is fully faithful, then any $h\colon y \to z$ is g-determined by Lemma 2.2.9.

The condition in Definition 4.2.1 can be replaced by an apparently weaker one:

Lemma 4.2.3. *In the notation of the previous definition, h is g-determined iff there exists a monic 2-cell ϵ such that $\operatorname{ran}_g hg = (h, \epsilon)$.*

Proof. The forwards implication is clear. Now let $\operatorname{ran}_g hg = (h, \epsilon)$ with ϵ monic, and let $\alpha, \beta\colon h \to h$ be the unique 2-cells such that $\epsilon \cdot \alpha g = \epsilon \cdot \epsilon$ and $\epsilon \cdot \beta g = 1_{hg}$. Since ϵ is monic, this implies that $\alpha g = \epsilon$. Then

$$\epsilon \cdot (\alpha \cdot \beta) g = \epsilon \cdot \alpha g \cdot \beta g = \epsilon \cdot \epsilon \cdot \beta g = \epsilon$$

and

$$\epsilon \cdot (\beta \cdot \alpha) g = \epsilon \cdot \beta g \cdot \alpha g = 1_{hg} \cdot \alpha g = \epsilon$$

imply that α and β are mutual inverses, since ϵ is the counit of a right extension along g. It follows that β is an isomorphism $(h, 1_{hg}) \cong (h, \epsilon)$ in $g^\circ{\downarrow}hg$, so $(h, 1_{hg})$ is also a right extension. □

Theorem 4.2.4. *Let $g\colon x \to y$ be a 1-cell, t be a monad on x, and s be a g-determined monad on y. If $g_\sharp t$ exists, then right extending along g gives a function*

$$\mathsf{Colax}_g(t, s) \to \mathsf{Mnd}_y(g_\sharp t, s)$$

natural in t and s. Moreover, it sends monic 2-cells to monic 2-cells.

Proof. Let ϵ be the counit of $g_\sharp t$. Given $(g, \psi) \in \mathsf{Colax}_g(t, s)$, we have $\psi\colon gt \to sg$. Let $\widehat{\psi} := \operatorname{ran}_g \psi\colon \operatorname{ran}_g gt \to \operatorname{ran}_g sg$. Since s is g-determined, we may take the right extension in the codomain to be $(s, 1_{sg})$, so that $\widehat{\psi}$ is the unique 2-cell such that

$$\widehat{\psi} g = \psi \cdot \epsilon. \tag{4.3}$$

We check that $\widehat{\psi}$ is a monad map $g_\sharp t \to s$. We have

$$(\widehat{\psi} \cdot \eta^{g_\sharp t}) g \overset{(4.3)}{=} \psi \cdot \epsilon \cdot \eta^{g_\sharp t} g \overset{(4.1)}{=} \psi \cdot g\eta^t \overset{(\psi.\eta)}{=} \eta^s g,$$

so $\widehat{\psi} \cdot \eta^{g_\sharp t} = \eta^s$, since they both have the same factorisation through the counit

1_{sg}. For the multiplication axiom we have

$$\begin{aligned}
(\widehat{\psi} \cdot \mu^{g_\sharp t})g &= \psi \cdot \epsilon \cdot \mu^{g_\sharp t} g && \text{by (4.3)} \\
&= \psi \cdot g\mu^t \cdot \epsilon t \cdot (g_\sharp t)\epsilon && \text{by (4.1)} \\
&= \mu^s g \cdot s\psi \cdot \psi t \cdot \epsilon t \cdot (g_\sharp t)\epsilon && \text{by } (\psi.\mu) \\
&= \mu^s g \cdot s\psi \cdot \widehat{\psi} gt \cdot (g_\sharp t)\epsilon && \text{by (4.3)} \\
&= \mu^s g \cdot \widehat{\psi} sg \cdot (g_\sharp t)\psi \cdot (g_\sharp t)\epsilon && \text{by the interchange law} \\
&= \mu^s g \cdot \widehat{\psi} sg \cdot (g_\sharp t)\widehat{\psi} g && \text{by (4.3)} \\
&= (\mu^s \cdot \widehat{\psi} s \cdot (g_\sharp t)\widehat{\psi})g.
\end{aligned}$$

For the same reason as before, this implies $\widehat{\psi} \cdot \mu^{g_\sharp t} = \mu^s \cdot \widehat{\psi} s \cdot (g_\sharp t)\widehat{\psi}$.

As in Theorem 4.1.5, naturality in t is with respect to monad maps in Mnd_x, and naturality in s is with respect to monad maps in Mnd_y between g-determined monads. In both cases, it follows from the fact that ran_g is a (partial) functor, and that the monads on y are g-determined.

The preservation of monic 2-cells is immediate from the fact that ran_g is a partial right adjoint. □

Remark 4.2.5. If $\mathcal{K}(x, x)$ has kernel pairs (e.g. if $\mathcal{K} = \mathsf{CAT}$ and x has kernel pairs), then a map of monads on x is monic iff it is monic as a 2-cell. This follows from Lemma 2.3.6, since a morphism is monic iff its kernel pair consists of identities.

Corollary 4.2.6. *In the notation of Theorem 4.2.4, if the counit of $g_\sharp t$ is invertible, then the function in that theorem is a bijection*

$$\mathsf{Colax}_g(t, s) \cong \mathsf{Mnd}_y(g_\sharp t, s).$$

Proof. Let ϵ be the counit of $g_\sharp t$. It is easy to see from Definition 2.3.3, that for an invertible 2-cell $\varphi\colon sg \to gt$ we have $(g, \varphi) \in \mathsf{Lax}(t, s)$ iff $(g, \varphi^{-1}) \in \mathsf{Colax}(t, s)$. Then $(g, \epsilon^{-1}) \in \mathsf{Colax}_g(t, g_\sharp t)$, and an inverse to the function in Theorem 4.2.4 is given by composing by (g, ϵ^{-1}) on the right in the 2-category $\mathsf{Colax} = \mathsf{MND}(\mathcal{K}^{\mathrm{op}})^{\mathrm{op}}$. □

This corollary gives a second universal property of pushforwards whose counit is an isomorphism. In $\mathcal{V}$-CAT, this is the case as soon as g is fully faithful and the right Kan extension defining $g_\sharp t$ is pointwise by Lemma 2.2.7.

We will find it useful to dualise some results in this and the previous section. Formally, these duals are obtained by passing to the 2-category $\mathcal{K}^{\mathrm{op}}$. This process only reverses the direction of the 1-cells, so monads in $\mathcal{K}^{\mathrm{op}}$ are the same thing as monads in $\mathcal{K}$. It does, however, interchange the concepts of lax and colax monad functors. Left and right extensions then become left and right lifts, respectively (Definition 2.2.1).

Definition 4.2.7. Let $g\colon x \to y$ be a 1-cell, and s be a monad on y. The **universal monad lift** of s along g is the right lift of sg along g, with its canonical monad structure. When it exists, we denote it by $g^\sharp s$.

Universal monad lifts have been used by Shirazi [55, §4] to give small presentations of several monads in probability theory as codensity monads. The use of the epithet 'universal', absent in the formally dual definition of pushforward, is required by the fact that monad lifts are a well-established concept, e.g. they are used in Theorem 2.3.14.

As with pushforwards, $g^\sharp$ is a functor $\mathsf{Mnd}_y \to \mathsf{Mnd}_x$ insofar as it is defined, which preserves those limits that $g^\circ \colon \mathcal{K}(y,y) \to \mathcal{K}(x,y)$ preserves. The dual of Theorem 4.1.5 shows that, given a 1-cell $g \colon x \to y$, and monads t on x and s on y, there is a natural bijection

$$\mathsf{Colax}_g(t,s) \cong \mathsf{Mnd}_x(t, g^\sharp s) \tag{4.4}$$

whenever $g^\sharp s$ exists. Moreover, the dual of Corollary 4.2.6 shows that, if additionally t is g-opdetermined and the counit of $g^\sharp s$ is an isomorphism, then there is a natural bijection

$$\mathsf{Lax}_g(t,s) \cong \mathsf{Mnd}_x(g^\sharp s, t). \tag{4.5}$$

A moment's glance at these isomorphisms together with their duals immediately gives two adjunctions between two pairs of full subcategories of Mnd_x and Mnd_y:

(i) Theorem 4.1.5 and (4.5) give $g^\sharp \dashv g_\sharp$; and

(ii) Corollary 4.2.6 and (4.4) give $g_\sharp \dashv g^\sharp$.

In each case, the domain and codomain of $g_\sharp$ must be suitable restricted to only include those monads which satisfy the assumptions of the results used to prove the adjunction. These conditions do not appear natural in a general 2-category. However, we will see in Section 4.3.2 that at least in the case of the first adjunction they are not rare when $\mathcal{K} = \mathsf{CAT}$.

4.3 Pushforwards in CAT

In this section we study pushforward monads in the 2-category CAT of locally small categories, functors and natural transformations. We give sufficient conditions for the existence of pushforwards, specialise and refine many of the results in the previous section, and show that codensity monads are invariant under free limit completions.

Much of the theory we develop here applies just as well to the 2-category $\mathcal{V}$-CAT for a Bénabou cosmos $\mathcal{V}$. We have chosen to write the results for CAT for ease of notation and terminology. In any case, the generalisation is straightforward.

4.3.1 Existence

Extensions in CAT are called Kan extensions, and there are well-known pointwise formulas that compute them in terms of (co)limits in the codomain category

(Example 2.2.4). In particular, if T is a monad on $\mathcal{A}$ and $G\colon \mathcal{A} \to \mathcal{B}$ is a functor, then the pushforward $G_\sharp T$ is given by

$$G_\sharp T(b) = \lim \left(b{\downarrow}G \xrightarrow{\Pi_b} \mathcal{A} \xrightarrow{T} \mathcal{A} \xrightarrow{G} \mathcal{B} \right), \tag{4.6}$$

for $b \in \mathcal{B}$ when the right-hand side exists. If $\mathcal{A}$ is small, then so is $b{\downarrow}G$, so that the limit in (4.6) is guaranteed to exist if $\mathcal{B}$ is complete. We can relax this condition by transferring the smallness hypothesis from $\mathcal{A}$ to G.

Definition 4.3.1. A functor $P\colon \mathcal{A} \to \mathsf{Set}$ is **small** if it is a small colimit of representables. A functor $G\colon \mathcal{A} \to \mathcal{B}$ is **representably small** if $\mathcal{B}(b, G-)\colon \mathcal{A} \to \mathsf{Set}$ is small for all $b \in \mathcal{B}$.

This terminology is taken from Day and Lack's work in [18], where they study limits in categories of small functors. Note that what we call representably small is what Avery and Leinster call corepresentably small in [8, Def. 4.6].

If a functor $P\colon \mathcal{A} \to \mathsf{Set}$ is representably small then it is small, since one can take b to be the singleton in the definition. The reverse implication does not hold, however; see [18, Ex. 8.1] for a counterexample. The following proposition gives a useful equivalent condition for a functor to be small. We say a category is **cofinally small** if it admits a cofinal functor from a small category.

Given a functor $P\colon \mathcal{A} \to \mathsf{Set}$, we write $\mathrm{El}(P)$ for its **category of elements**, which is the comma category $1{\downarrow}P$ where $1 \in \mathsf{Set}$ is the singleton.

Proposition 4.3.2. *The following are equivalent for a functor* $P\colon \mathcal{A} \to \mathsf{Set}$*:*

(i) P *is small;*

(ii) $P = \mathrm{Lan}_K H$ *for some functors* $K\colon \mathcal{B} \to \mathcal{A}$ *and* $H\colon \mathcal{B} \to \mathsf{Set}$ *with* $\mathcal{B}$ *small;*

(iii) *the category of elements* $\mathrm{El}(P)$ *of* P *is cofinally small.*

Proof. The equivalence between (i) and (ii) is proved in the more general enriched context in Kelly's book [33, Prop. 4.83].

Now assume (ii), and let $\eta\colon H \to PK$ be the unit of the left Kan extension. Then η induces a functor $\mathrm{El}(H) \to \mathrm{El}(PK)$, which we can compose with the obvious functor $\mathrm{El}(PK) \to \mathrm{El}(P)$ to get a functor $D\colon \mathrm{El}(H) \to \mathrm{El}(P)$. It suffices to show that D is cofinal, i.e. that the comma category $D{\downarrow}(a, x)$ is connected for every $a \in \mathcal{A}$ and $x \in Pa$.

Since $\mathcal{B}$ is small and Set is cocomplete, the left Kan extension is pointwise, so the assignment

$$Kb \xrightarrow{f} a \qquad \longmapsto \qquad Hb \xrightarrow{\eta_b} PKb \xrightarrow{Pf} Pa$$

forms a colimiting cocone for $K{\downarrow}a \xrightarrow{\Pi_a} \mathcal{B} \xrightarrow{H} \mathsf{Set}$. Hence, there exists some $f\colon Kb \to a$ and $y \in Hb$ such that $(Pf \cdot \eta_b)(y) = x$, and so f is a morphism $D(b, y) \to (a, x)$ in $\mathrm{El}(P)$. Moreover, for any other $f'\colon D(b', y') \to (a, x)$, there must be a zigzag of morphisms in $\mathrm{El}(H\Pi_a)$ connecting (b, y) and (b', y'), by the computation of colimits in Set. The same zigzag connects them in $D{\downarrow}x$, so this category is connected.

Lastly, we show that (iii) implies (i). One of the consequences of the Yoneda lemma is that P is the colimit of the composite $\mathrm{El}(P)^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}} \to [\mathcal{A}, \mathsf{Set}]$. If $\mathrm{El}(P)$ admits a cofinal functor from a small category, then this colimit is equivalent to a small colimit, showing that P is small. □

Remark 4.3.3. Despite what the name might suggest, a subfunctor of a small functor need not be small. Let $\mathcal{S}$ be a large discrete category, and $\mathcal{S}_0$ be the category resulting from freely adjoining an initial object, denoted by 0, to $\mathcal{S}$. Then the functor $G\colon \mathcal{S}_0 \to \mathsf{Set}$ which is constant at the singleton set is small (in fact representable), but the subfunctor F of G which sends 0 to $\varnothing$ and acts as G otherwise is not small. This is easily seen, by the previous proposition, from the fact that $\mathrm{El}(F)$ is isomorphic to $\mathcal{S}$, which is clearly not cofinally small.

Examples 4.3.4.

(i) A functor $\mathcal{A} \to \mathsf{Set}$ from an (essentially) small category is small by taking $K = 1_{\mathcal{A}}$ in Proposition 4.3.2(ii). Consequently, any functor $\mathcal{A} \to \mathcal{B}$ is representably small.

(ii) A right adjoint $G\colon \mathcal{A} \to \mathcal{B}$ is representably small, since $\mathcal{B}(b, G-) \cong \mathcal{A}(Fb, -)$ is already representable, where $F \dashv G$.

(iii) The fully faithful functor $D\colon \mathsf{Set} \hookrightarrow \mathsf{Top}$ that equips a set with the discrete topology is representably small. Indeed, given a topological space X, the full subcategory $\mathcal{S}$ of $X{\downarrow}D = \mathrm{El}(\mathsf{Top}(X, D-))$ on the surjections is cofinal. This follows easily from the facts that Top has an (epi, strong mono) factorisation system and that subspaces of discrete spaces are discrete. Note that $\mathcal{S}$ is small, since for any surjection $X \to DS$ we must have $|S| \leq |X|$.

(iv) The fully faithful forgetful functor $V\colon \mathsf{Field} \hookrightarrow \mathsf{CRing}$ is representably small. Given a ring R and a field k, any homomorphism $f\colon R \to k$ factors through a unique residue field of R, namely $\mathrm{Frac}(R/\mathfrak{p})$, where $\mathfrak{p} = \ker f$. Hence, $R{\downarrow}V$ has connected components in bijection with $\operatorname{Spec} R$, and each of those components has an initial object.

(v) The functor $(-) + E\colon \mathsf{Set} \to \mathsf{Set}$ for E any set is representably small. This is the underlying functor of the exception monad of Example 2.4.12, so we denote it by E_E. Given a set X, the connected components of $X{\downarrow}\mathsf{E}_E$ are in bijection with the functions $f\colon X \to 1 + E$, and the component corresponding to f has an initial object given by the function $X \to f^{-1}1 + E$ which acts as the identity on $f^{-1}1$ and as f otherwise.

(vi) Let K be a set, and $\mathsf{Set}_{\geq K}$ be the full subcategory of Set on the sets of cardinality at least that of K. The inclusion $J\colon \mathsf{Set}_{\geq K} \hookrightarrow \mathsf{Set}$ is representably small. Indeed, let X be a set. If $|X| \geq |K|$, then $X{\downarrow}J$ has an initial object: the identity on X. Otherwise, by cardinal comparability, there exists an injection $m\colon X \to K$. We claim that the subcategory $\mathcal{S}$ of $X{\downarrow}J$ depicted by

$$\begin{tikzcd}
X \arrow[d, "m"'] \arrow[dr, "m'"] & \\
K \arrow[r, shift left, "n_1"] \arrow[r, shift right, "n_2"'] & K',
\end{tikzcd}$$

where (n_1, n_2) is the cokernel pair of m, and $m' = n_1 m = n_2 m$, is cofinal.

First note that n_1 and n_2 are also injective, so $K' \in \mathsf{Set}_{\geq K}$. Now take any $f\colon X \to Y$ in $X{\downarrow}J$. It factors through m, either because $X = \varnothing$ or because m splits, so $\mathcal{S}{\downarrow}f$ is nonempty. Seeing that it is connected involves a straightforward case analysis. We illustrate the most complicated case: where f factors as gm' and hm'. The next diagram shows a zigzag connecting g and h in $\mathcal{S}{\downarrow}f$:

$$\begin{tikzcd}
 & K' \arrow[ddr, bend left, "g"] & \\
 & K \arrow[u, "n_1"] \arrow[d, "n_2"'] & \\
X \arrow[uur, bend left, "m'"] \arrow[ur, "m"] \arrow[r, "m'"] \arrow[dr, "m"'] \arrow[ddr, bend right, "m'"'] & K' \arrow[r, "{(hn_2, gn_1)}"] & Y \\
 & K \arrow[u, "n_1"] \arrow[d, "n_2"'] & \\
 & K' \arrow[uur, bend right, "h"'] &
\end{tikzcd}$$

where (hn_2, gn_1) is the unique function $K' \to Y$ whose composites with n_1 and n_2 are hn_2 and gn_1, respectively.

The importance of representably small functors in the theory of pushforward monads in CAT is clear from the next proposition.

Proposition 4.3.5. *Let $G\colon \mathcal{A} \to \mathcal{B}$ be a representably small functor into a complete category. Then $G_\sharp T$ exists for any monad T on $\mathcal{A}$.*

Proof. It suffices to show that the right Kan extension along G of any functor $F\colon \mathcal{A} \to \mathcal{C}$ with $\mathcal{C}$ complete exists. This follows from the limit formula (2.5), since $\mathcal{C}$ is complete and $b{\downarrow}G = \mathrm{El}(\mathcal{B}(b, G-))$ is cofinally small by assumption. $\square$

It follows that one can take pushforwards along each of the functors listed in Examples 4.3.4. Representably small functors enjoy many convenient properties. For example, they are closed under composition, as proved by Avery and Leinster [8, Lem. 4.7]. With this result, we can show that the pushforward construction can often be iterated.

Proposition 4.3.6. *Let $G\colon \mathcal{A} \to \mathcal{B}$ be a representably small functor into a complete category, and T be a monad on $\mathcal{A}$. If $\mathcal{A}^T$ has finite connected colimits, then* *the top functor in the square*

$$\begin{tikzcd}
\mathcal{A}^T \arrow[r, "G'"] \arrow[d, "U^T"'] & \mathcal{B}^{G_\sharp T} \arrow[d, "U^{G_\sharp T}"] \\
\mathcal{A} \arrow[r, "G"'] & \mathcal{B},
\end{tikzcd}$$

corresponding under Theorem 2.3.7 to the counit of $G_\sharp T$, is also a representably small functor into a complete category.

Proof. That $\mathcal{B}^{G_\sharp T}$ is complete follows from the fact that $U^{G_\sharp T}$ creates limits. By Example 4.3.4(ii) and the fact that the composite of two representably small functors is representably small, $GU^T = U^{G_\sharp T}G'$ is representably small, so $\mathcal{B}(b, U^{G_\sharp T}G'-) \cong \mathcal{B}^{G_\sharp T}(F^{G_\sharp T}b, G'-)$ is small for every $b \in \mathcal{B}$. We can now use the theory of limits in categories of small presheaves to show that $\mathcal{B}^{G_\sharp T}(b, G'-)$ is small for every $b \in \mathcal{B}^{G_\sharp T}$, not just the free $G_\sharp T$-algebras.

Let $[\mathcal{A}^T, \mathsf{Set}]_\mathrm{s}$ denote the full subcategory of $[\mathcal{A}^T, \mathsf{Set}]$ on the small functors. Day and Lack's [18, Prop. 4.3] shows that $[\mathcal{A}^T, \mathsf{Set}]_\mathrm{s}$ has all finite connected limits and, as the Yoneda embedding $(\mathcal{A}^T)^{\mathrm{op}} \hookrightarrow [\mathcal{A}^T, \mathsf{Set}]$ factors through $[\mathcal{A}^T, \mathsf{Set}]_\mathrm{s}$, it is easy to see that such limits are computed objectwise. Every $b \in \mathcal{B}^{G_\sharp T}$ is a reflexive coequaliser of free $G_\sharp T$-algebras, say of the pair of morphisms (f, g). Since equalisers are finite connected limits, we can take the equaliser E of $\mathcal{B}^{G_\sharp T}(f, G'-)$ and $\mathcal{B}^{G_\sharp T}(g, G'-)$ in $[\mathcal{A}^T, \mathsf{Set}]_\mathrm{s}$, which satisfies

$$Ea \cong \mathcal{B}^{G_\sharp T}(\mathrm{coeq}(f, g), G'a) \cong \mathcal{B}^{G_\sharp T}(b, G'a),$$

naturally in $a \in \mathcal{A}^T$. Hence, $\mathcal{B}^{G_\sharp T}(b, G'-) \cong E$ is small. ☐

Remark 4.3.7. It seems that the hypothesis about the existence of certain colimits in $\mathcal{A}^T$ cannot be dropped if one wants to conclude that G' is representably small in general. This is because a composite GF being representably small does not imply that F is representably small, even when G is faithful. (The author has not found a counterexample where G is merely monadic.)

For an example, let $\mathcal{S}_0$ be as in Remark 4.3.3, and let Ω denote the category with two objects, a and b, and exactly one non-identity morphism $a \to b$. The functor $F\colon \mathcal{S}_0 \to \Omega$ which sends 0 to a and the rest of $\mathcal{S}$ to b is not representably small, since $\mathcal{S}_0(b, F-)$ is not small by Remark 4.3.3. Now let G be the unique functor $2 \to 1$, which is faithful. As $1(*, GF-)$ is constant at the singleton (where $*$ is the unique object of 1), GF is representably small by the same remark.

Example 4.3.8. Of course, another way of having G' in Proposition 4.3.6 be representably small is simply having $\mathcal{A}$ small. Taking T to be the identity monad, this gives a codensity version of monadic towers. These (or rather their comonadic dual) were introduced by Appelgate and Tierney in [5], where they are used to decompose an adjunction into a reflection followed by a number of monadic adjunctions. In the codensity case, we need not start with an adjunction, a functor suffices; and instead of a reflection, the process ends once we reach a codense functor (Definition 2.2.11). Here are some examples (see Figure 4.1):

(i) Let $2\colon \mathsf{1} \to \mathsf{Set}$ be the functor which picks out a two-element set. It follows from Examples 2.4.15 and 4.1.3(iii) that the codensity monad of 2 is the double powerset monad. Its category of algebras is that of complete atomic Boolean algebras, which is equivalent to $\mathsf{Set}^{\mathrm{op}}$. The functor $2'\colon \mathsf{1} \to \mathsf{Set}^{\mathrm{op}}$ picks out 1, because its composite with $2^{-}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ is 2. Since 1 is a dense generator of Set, the functor $2'$ is codense.

(ii) Let $I\colon \mathsf{fdVect} \hookrightarrow \mathsf{Vect}$ be the inclusion of the full subcategory of finite dimensional vector spaces over a fixed field. As proved by Leinster [45, §7], its codensity monad is the double dualisation monad, whose category of algebras is equivalent to $\mathsf{Vect}^{\mathrm{op}}$. The functor $I'\colon \mathsf{fdVect} \to \mathsf{Vect}^{\mathrm{op}}$ is the composite of I^{op} and the equivalence $(-)^*\colon \mathsf{fdVect} \to \mathsf{fdVect}^{\mathrm{op}}$ given by the fact that finite dimensional vector spaces are self-dual. Now I is dense, because it is the inclusion of the category of finitely presentable objects of a locally finitely presentable category, so I' is codense.

(iii) Let $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$ be the inclusion of the full subcategory of finite sets. Its codensity monad is the ultrafilter monad on Set, whose category of algebras is CHaus. The functor $J'\colon \mathsf{FinSet} \to \mathsf{CHaus}$ equips a finite set with its unique compact Hausdorff topology: the discrete topology. Its codensity monad $J'_\sharp 1$ sends a space X to the set of ultrafilters in its Boolean algebra of clopen subsets. The category of $J'_\sharp 1$-algebras is equivalent to Stone, the category of Stone spaces. This follows from the following argument outlined, by Sipoș in [57, §5]:

Note that J' factors as $\mathsf{FinSet} \xrightarrow{K} \mathsf{Stone} \xrightarrow{U} \mathsf{CHaus}$. Under Stone duality, K^{op} corresponds to the inclusion of the category of finite Boolean algebras into the category of all Boolean algebras. The former category contains all of the finitely generated free Boolean algebras. Since Boolean algebras are the algebras of a finitary algebraic theory, it follows that K^{op} is dense, and so K is codense. By Remark 4.1.8, since U is a right adjoint, we have

$$J'_\sharp 1 = (UK)_\sharp 1 = U_\sharp(K_\sharp 1) = U_\sharp 1.$$

As U is already monadic (being the inclusion of a reflective subcategory), the category of algebras of $U_\sharp 1$ is Stone. It also follows that $J'' = K$, which is codense, so the monadic tower stabilises after two steps.

4.3.2 An adjunction between categories of monads

We now revisit the adjunctions found at the end of Section 4.2. One of them takes a particularly nice form when one pushes forwards along a fully faithful functor (Theorem 4.3.11). Recall that we write $\mathrm{rift}_G F$ and $\mathrm{lift}_G F$ for the right and left Kan lifts of F through G (Definition 2.2.1).

$$\begin{tikzcd}
 & \mathsf{Set}^{\mathrm{op}} \arrow[d, "2^{-}"] \\
1 \arrow[ur, "2'"] \arrow[r, "2"'] & \mathsf{Set}
\end{tikzcd}$$

(i)

$$\begin{tikzcd}
 & \mathsf{Vect}^{\mathrm{op}} \arrow[d, "(-)^{*}"] \\
\mathsf{fdVect} \arrow[ur, "I'"] \arrow[r, "I"'] & \mathsf{Vect}
\end{tikzcd}$$

(ii)

$$\begin{tikzcd}
 & \mathsf{Stone} \arrow[d, "U"] \\
 & \mathsf{CHaus} \arrow[d, "V"] \\
\mathsf{FinSet} \arrow[uur, bend left, "J'' = K"] \arrow[ur, "J'"] \arrow[r, "J"'] & \mathsf{Set}
\end{tikzcd}$$

(iii)

Figure 4.1: Three examples of codensity monadic towers.

Lemma 4.3.9. *Let θ be a natural isomorphism fitting into a diagram*

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dr, "G"] \arrow[d, phantom, "\theta \Uparrow"] & \\
\mathcal{A} \arrow[ur, "H"] \arrow[rr, "F"'] & {} & \mathcal{C}
\end{tikzcd}$$

where G is fully faithful. Then $\mathrm{rift}_G\, F = (H, \theta)$ *and* $\mathrm{lift}_G\, F = (H, \theta^{-1})$.

Proof. We prove the statement about the right Kan lift; the other is dual. Note that since G is fully faithful, so is $G_\circ \colon [\mathcal{A}, \mathcal{B}] \to [\mathcal{A}, \mathcal{C}]$. Given any $\alpha \colon F \to GH'$, it follows that there is a unique $\widehat{\alpha} \colon H \to H'$ such that $G\widehat{\alpha} = \alpha \cdot \theta^{-1}$. This is exactly what it means for (H, θ) to be the right Kan lift of F through G. ☐

We aim to use the dual of Corollary 4.2.6, as given in (4.5). Lemma 4.3.9 allows us to simplify the conditions on the corresponding monads. Let $G \colon \mathcal{A} \to \mathcal{B}$ be fully faithful. The previous lemma, together with the dual of Lemma 4.2.3, implies that any monad on $\mathcal{A}$ is G-opdetermined (see Definition 4.2.1). Moreover, a monad S on $\mathcal{B}$ such that $G^\sharp S$ exists and has an isomorphism as counit is precisely one that **restricts along** G, i.e. such that there exist an endofunctor S' of $\mathcal{A}$ (which inherits a monad structure) and an isomorphism $GS' \cong SG$, since Lemma 4.3.9 then implies that $G^\sharp S \cong S'$.

Examples 4.3.10.

(i) Clearly, the identity monad restricts along any fully faithful functor.

(ii) Most of the monads on Set introduced in Section 2.4 restrict along the inclusion FinSet ↪ Set. For example: the E-exception monad for a finite set E, the free M-set monad for a finite monoid M, the powerset monad, the endomorphism monad of any finite set (e.g. the double powerset monad), the filter monad and the ultrafilter monad.

Theorem 4.3.11. *Let $G \colon \mathcal{A} \to \mathcal{B}$ be a fully faithful, representably small functor into a complete category. There is an adjunction*

$$\begin{tikzcd}
\mathsf{Mnd}_{\mathcal{A}} \arrow[r, bend right, "G_\sharp"'] \arrow[r, phantom, "\perp"] & \mathsf{Mnd}_{\mathcal{B}}^{\mathrm{res}G} \arrow[l, bend right, "G^\sharp"']
\end{tikzcd}$$

where $\mathsf{Mnd}_{\mathcal{B}}^{\mathrm{res}G}$ *is the full subcategory of* $\mathsf{Mnd}_{\mathcal{B}}$ *on the monads that restrict along* G*. Moreover, the right adjoint* $G_\sharp$ *is fully faithful.*

Proof. Since G is representably small and $\mathcal{B}$ is complete, $G_\sharp$ is defined pointwise on all $\mathsf{Mnd}_{\mathcal{A}}$. Since G is fully faithful, the counit of a right Kan extension along it is an isomorphism, showing that $G_\sharp$ has the right codomain. Lemma 4.3.9 ensures that $G^\sharp$ is defined on all $\mathsf{Mnd}_{\mathcal{B}}^{\mathrm{res}G}$. By Theorem 4.1.5 and the dual of Theorem 4.2.4, there is a bijection

$$\mathsf{Mnd}_{\mathcal{A}}(G^\sharp S, T) \cong \mathsf{Lax}_G(T, S) \cong \mathsf{Mnd}_{\mathcal{B}}(S, G_\sharp T)$$

natural in $S \in \mathsf{Mnd}_{\mathcal{B}}^{\mathrm{res}G}$ and $T \in \mathsf{Mnd}_{\mathcal{A}}$.

As G is fully faithful, so are Ran_G and $G_\circ$, and hence so is $G_\sharp$. This can also be easily seen from the fact that the counit of the adjunction is an isomorphism. □

Remark 4.3.12. A consequence of the counit $\epsilon\colon (G_\sharp T)G \to GT$ being invertible when G is fully faithful is that (G, ϵ^{-1}) is a colax monad functor $(\mathcal{B}, G_\sharp T) \to (\mathcal{A}, T)$. Hence, not only is there a lift of G to the respective categories of algebras, but also an extension to the Kleisli categories. Explicitly, we have two commuting squares of functors:

$$\begin{tikzcd}
\mathcal{A}^T \arrow[r, "G^\epsilon"] \arrow[d, "U^T"'] & \mathcal{B}^{G_\sharp T} \arrow[d, "U^{G_\sharp T}"] \\
\mathcal{A} \arrow[r, hook, "G"'] & \mathcal{B}
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\mathcal{A} \arrow[r, hook, "G"] \arrow[d, "F_T"'] & \mathcal{B} \arrow[d, "F_{G_\sharp T}"] \\
\mathcal{A}_T \arrow[r, "G_{\epsilon^{-1}}"'] & \mathcal{B}_{G_\sharp T}.
\end{tikzcd}$$

Given a T-algebra $\alpha\colon Ta \to a$, the functor G^ϵ produces the $G_\sharp T$-algebra

$$(G_\sharp T)Ga \xrightarrow{\epsilon_a} GTa \xrightarrow{G\alpha} Ga.$$

Given a morphism $f\colon a \to Tb$ in $\mathcal{A}$, the functor $G_{\epsilon^{-1}}$ gives the morphism

$$Ga \xrightarrow{Gf} GTb \xrightarrow{\epsilon_b^{-1}} (G_\sharp T)Gb.$$

In fact, it follows from the fact that G is fully faithful and ϵ is invertible that G^ϵ and $G_{\epsilon^{-1}}$ are also fully faithful. Indeed, let $g\colon G^\epsilon(a,\alpha) \to G^\epsilon(b,\beta)$ be a $G_\sharp T$-algebra map. Since G is fully faithful, $g = Gf$ for a unique $f\colon a \to b$. Now consider the diagram

$$\begin{tikzcd}
(G_\sharp T)Ga \arrow[r, "\epsilon_a"] \arrow[d, "(G_\sharp T)Gf"'] & GTa \arrow[r, "G\alpha"] \arrow[d, "GTf"'] & Ga \arrow[d, "Gf"] \\
(G_\sharp T)Gb \arrow[r, "\epsilon_b"'] & GTb \arrow[r, "G\beta"'] & Gb.
\end{tikzcd}$$

The outside face commutes by assumption, and the left square commutes by naturality of ϵ. Since ϵ_a is epic, it follows that the right square also commutes, so that f is a T-algebra map $(a,\alpha) \to (b,\beta)$. To see that $G_{\epsilon^{-1}}$ is fully faithful, simply note that for any morphism $g\colon Ga \to (G_\sharp T)Gb$ the composite $\epsilon_b \cdot g$ is in the image of G, and hence g is in the image of $G_{\epsilon^{-1}}$.

Example 4.3.13. The only endofunctor of CHaus which fixes underlying sets is the identity. Indeed, such an endofunctor amounts to an endomorphism of the ultrafilter monad on Set by Theorem 2.3.7, which is the codensity monad of $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$. Since $J_\sharp\colon \mathsf{Mnd}_{\mathsf{FinSet}} \to \mathsf{Mnd}_{\mathsf{Set}}$ is fully faithful, this corresponds to an endomorphism of the identity monad on FinSet, which can only be the identity.

Example 4.3.14. The adjunction of Theorem 4.3.11 takes a particularly simple form when we take G to be $J\colon \mathsf{Set}_{\geq K} \hookrightarrow \mathsf{Set}$ from Example 4.3.4(vi). All but two monads on Set have monic unit by Lemma 2.4.8; i.e. the inconsistent monads of Examples 2.4.9 and 2.4.10.

Every consistent monad T on Set restricts along J. Moreover, for every set X such that TX is nonempty, we have $(J_\sharp J^\sharp T)X \cong TX$. This is clear if $|X| \geq |K|$. Otherwise, Example 4.3.4(vi) shows that $(J_\sharp J^\sharp T)X$ is the equaliser of

$$\begin{tikzcd} TK \arrow[r, shift left, "Tn_1"] \arrow[r, shift right, "Tn_2"'] & TK', \end{tikzcd}$$

where (n_1, n_2) is the cokernel pair of an injection $m\colon X \to K$. Since TX is nonempty, there is a retraction r of Tm. We also have the composite

$$s = TK' \xrightarrow{T(\eta^T_K,\, Tm \cdot r \cdot \eta^T_K)} T^2K \xrightarrow{\mu^T_K} TK,$$

where (f_1, f_2) denotes the unique morphism such that $(f_1, f_2) \cdot n_i = f_i$ for $i \in \{1, 2\}$, for f_1 and f_2 such that $f_1 m = f_2 m$. One checks that r and s, together with Tm, Tn_1 and Tn_2 form a split equaliser diagram, showing that $(J_\sharp J^\sharp T)X \cong TX$.

If we take $K = 1$, then even the inconsistent monads restrict along J, giving a reflection

$$\begin{tikzcd} \mathsf{Mnd}_{\mathsf{Set}_{\geq 1}} \arrow[r, hook, bend right, "J_\sharp"'] \arrow[r, phantom, "\perp"] & \mathsf{Mnd}_{\mathsf{Set}} \arrow[l, bend right, "J^\sharp"'] \end{tikzcd} \tag{4.7}$$

Let a and b be the two functions $1 \to 2$. Given a monad T on Set, the equaliser of the pair (Ta, Tb) is the set of **pseudoconstants** of T. A pseudoconstant can be understood as a unary term in the theory of T which takes a constant value in every nonempty algebra. Every constant induces a pseudoconstant; an example of a pseudoconstant that does not come from a constant is the unique element of any nonempty model of the theory of sets where all elements are equal.

The monad on $\mathsf{Mnd}_{\mathsf{Set}}$ induced by the reflection (4.7) sends a monad T to $J_\sharp J^\sharp T$, which agrees with T on all nonempty sets and which sends $\varnothing$ to the set of pseudoconstants of T. In other words, $J_\sharp J^\sharp$ realises the pseudoconstants of a monad as actual constants. The argument above implies the well-known fact that as soon as there is at least one constant (so that $TX \neq \varnothing$ for all X), all pseudoconstants are induced by constants.

As outlined at the end of Section 4.2, there is also a dual partial adjunction. If $G\colon \mathcal{A} \to \mathcal{B}$ is fully faithful and representably small and $\mathcal{B}$ is complete, then $G_\sharp T$ is a G-determined monad for all monads T on $\mathcal{A}$. This gives:

$$\begin{tikzcd}
\mathsf{Mnd}_{\mathcal{A}} \arrow[r, hook, bend left, "G_\sharp"] \arrow[r, phantom, "\perp"] & \mathsf{Mnd}_{\mathcal{B}}^{G\text{det}} \arrow[l, dashed, bend left, "G^\sharp"]
\end{tikzcd}$$

where $\mathsf{Mnd}_{\mathcal{B}}^{G\text{det}}$ is the full subcategory of $\mathsf{Mnd}_{\mathcal{B}}$ on the G-determined monads. Notice the interchange of the left and right adjoints compared to Theorem 4.3.11. In this case, $G^\sharp$ need not be defined on all of $\mathsf{Mnd}_{\mathcal{B}}^{G\text{det}}$. A simple sufficient condition to make this adjunction total is that G be a left adjoint (such as the discrete-topology functor $\mathsf{Set} \to \mathsf{Top}$), since then right Kan lifts through G always exist by the dual of Lemma 2.2.3(iii).

Examples 4.3.15.

(i) If $G\colon \mathcal{A} \to \mathcal{B}$ is fully faithful, then any right Kan extension along G is G-determined by Lemma 4.2.3. For example, the endomorphism monad on Set of a finite set X is J-determined, where $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$. This follows from Example 4.1.3(iii), because it is the codensity monad of $1 \xrightarrow{X} \mathsf{FinSet} \xrightarrow{J} \mathsf{Set}$, and hence is given by $\mathrm{Ran}_{JX}\, JX \cong \mathrm{Ran}_J(\mathrm{Ran}_X\, JX)$.

In particular, the double-powerset monad on Set is J-determined. This is in stark contrast to the powerset monad P on Set, which is not J-determined. If it were, then $\mathsf{P} \cong \mathrm{Ran}_J\, \mathsf{P}J$, but we will see in Theorem 6.1.14 below that the right-hand side is the filter monad.

(ii) Let $G\colon \mathcal{A} \hookrightarrow \mathcal{B}$ be the inclusion of a full subcategory. By definition, a monad on $\mathcal{B}$ is (pointwise) G-determined precisely when it has the *limit property* of Adámek and Sousa [3, Def. 6.2]. Assuming $\mathcal{A}$ is small and $\mathcal{B}$ is complete, their Theorem 6.5 shows that the codensity monad of G is the smallest G-determined submonad of a carefully chosen G-determined monad S on $\mathcal{B}$. One of the key properties of S is that η^S is pointwise monic, and hence so is $\eta^S G$ since G is fully faithful.

This result can be generalised easily using our theory of pushforwards. Let $G\colon \mathcal{A} \to \mathcal{B}$ be a functor such that $G_\sharp 1$ exists. If T is a G-determined monad on $\mathcal{B}$ such that $\eta^T G$ is a monic natural transformation, then $G_\sharp 1$ is the smallest G-determined submonad of T.

This follows from the fact that for any monad T on $\mathcal{B}$, the pair $(G, \eta^T G)$ is a colax monad functor $1_{\mathcal{A}} \to T$. If T is G-determined and $\eta^T G$ is monic, then Theorem 4.2.4 gives a monic monad map $\widehat{\eta^T G}\colon G_\sharp 1 \to T$. If $\theta\colon S \to T$ is a G-determined submonad, then the naturality in T of this construction ensures that $\theta \cdot \widehat{\eta^S G} = \widehat{\eta^T G}$, making $G_\sharp 1$ the smallest G-determined submonad of T.

Remark 4.3.16. One of the main consequences of Theorem 4.3.11 is that, under some completeness conditions, the category of monads on a full subcategory $\mathcal{A}$ of

$\mathcal{B}$ is a full subcategory of $\mathsf{Mnd}_{\mathcal{B}}$. In fact, we can obtain an alternative fully faithful functor $\mathsf{Mnd}_{\mathcal{A}} \hookrightarrow \mathsf{Mnd}_{\mathcal{B}}$ whenever $G \colon \mathcal{A} \to \mathcal{B}$ exhibits $\mathcal{B}$ as the free cocompletion of a small $\mathcal{A}$ under a class of small colimit shapes (or weights in the enriched case). In this case, the assignment $T \mapsto \mathrm{Lan}_G\, GT$ extends to the required functor.

This follows from the theory of relative monads of Altenkirch, Chapman and Uustalu [4]. In Theorems 4.6 and 4.7, they show that if G is *well-behaved* then Lan_G extends to a functor (necessarily fully faithful) from the category RMnd_G of G-relative monads to $\mathsf{Mnd}_{\mathcal{B}}$. Arkor and McDermott [6, Rmk. 4.31] observed that G is *well-behaved* in their sense iff it exhibits $\mathcal{B}$ as a free cocompletion of $\mathcal{A}$ as above. At the same time, it is easy to see that if T is a monad on $\mathcal{A}$ and G is fully faithful then GT is a G-relative monad, and that this extends to a fully faithful functor $\mathsf{Mnd}_{\mathcal{A}} \hookrightarrow \mathsf{RMnd}_G$. Altogether, we get a fully faithful functor

$$G_\flat \colon \mathsf{Mnd}_{\mathcal{A}} \xhookrightarrow{G_\circ} \mathsf{RMnd}_G \xhookrightarrow{\mathrm{Lan}_G} \mathsf{Mnd}_{\mathcal{B}}.$$

Just like $G_\sharp$, this functor lands in the full subcategory $\mathsf{Mnd}_{\mathcal{B}}^{\mathrm{res}G}$, and it is easy to see from the definition of Lan_G that it provides a left adjoint to the restriction functor $G^\sharp$. Hence, when G is a free cocompletion as above and $\mathcal{B}$ is complete, we have

$$\begin{tikzcd}
\mathsf{Mnd}_{\mathcal{A}} \arrow[r, hook, bend left=60, "G_\flat"] \arrow[r, hook, bend right=60, "G_\sharp"'] \arrow[r, phantom, shift left=3ex, "\perp"] \arrow[r, phantom, shift right=3ex, "\perp"] & \mathsf{Mnd}_{\mathcal{B}}^{\mathrm{res}G}. \arrow[l, "G^\sharp" description]
\end{tikzcd}$$

A prime example of this is given by taking G to be the inclusion $J \colon \mathcal{A}_{\mathrm{fp}} \to \mathcal{A}$ of the full subcategory of finitely presentable objects of a locally finitely presentable category. In this case, $G_\flat T$ is the unique finitary extension of a monad T on $\mathcal{A}_{\mathrm{fp}}$ to a monad on $\mathcal{A}$, so that $G_\sharp T$ is finitary iff it coincides with $G_\flat T$.

One of the main examples we consider below (Section 6.1) is of this form: the inclusion $J \colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$. We will see that, for any consistent monad T on FinSet, $J_\sharp T$ does not coincide with $J_\flat T$, and in fact it has no rank (Theorem 6.1.24).

Note that there are monads on Set which restrict to FinSet and yet are not in the image of $J_\flat$ or $J_\sharp$. An example is the powerset monad P. Clearly, $J_\flat J^\sharp \mathsf{P}$ is the finite powerset monad, while $J_\sharp J^\sharp \mathsf{P}$ is the filter monad by Theorem 6.1.14 below.

4.3.3 Limit completions and codensity

We finish this section with the observation that codensity monads are invariant under free limit completions, in a suitable sense. This allows us to relate codensity monads to Diers's multiadjunctions [23, p. 58] and Tholen's pro-adjunctions [61, p. 148]. In particular, Diers's results allow us to identify the category of algebras of the codensity monad of $\mathsf{Field} \hookrightarrow \mathsf{CRing}$ as the free product completion of Field (Example 4.3.23(i)).

We will borrow the notation for cocompletions under classes of colimits from the last part of Section 2.2 and apply it to completions under classes of limits instead. Explicitly, if $\mathfrak{D}$ is a class of diagrams in a category $\mathcal{A}$, we write $\mathfrak{D}[\mathcal{A}]$ for the result of freely adjoining limits for each diagram in $\mathfrak{D}$. If $\mathfrak{C}$ is a class of categories, thought of as shapes of diagrams, we write $\mathfrak{C}(\mathcal{A})$ for the free completion of $\mathcal{A}$ under $\mathfrak{C}$-limits.

Suppose we are interested in the monadic reflection, and hence in the codensity monad $G_\sharp 1$, of a functor $G\colon \mathcal{A} \to \mathcal{B}$. This will come equipped with a comparison functor $K\colon \mathcal{A} \to \mathcal{B}^{G_\sharp 1}$ over $\mathcal{B}$. In particular, if $D\colon \mathcal{I} \to \mathcal{A}$ is a diagram such that GD has a limit, then KD also has a limit, created by $U^{G_\sharp 1}$ from $\lim GD$. Let $\mathfrak{D}$ be a class of diagrams $D\colon \mathcal{I} \to \mathcal{A}$ such that $\lim GD$ exists. We could instead consider the codensity monad of the induced functor $\widehat{G}\colon \mathfrak{D}[\mathcal{A}] \to \mathcal{B}$. Since monadic functors create limits, one would expect these two monads to be related. Indeed, they are the same:

Proposition 4.3.17. *Let $G\colon \mathcal{A} \to \mathcal{B}$ be a functor with a pointwise codensity monad and $\mathfrak{D}$ be a class of diagrams $D\colon \mathcal{I} \to \mathcal{A}$ such that* $\lim GD$ *exists. Consider the commutative diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "G"] \arrow[dr, hook, "J"'] & & \mathcal{B}. \\
& \mathfrak{D}[\mathcal{A}] \arrow[ur, "\widehat{G}"'] &
\end{tikzcd}$$

Then G and $\widehat{G}$ have the same codensity monad.

Proof. It suffices to show that $\widehat{G}$ preserves $\operatorname{Ran}_J J$, since then

$$\operatorname{Ran}_G G = \operatorname{Ran}_{\widehat{G}J} \widehat{G}J \cong \operatorname{Ran}_{\widehat{G}}(\operatorname{Ran}_J \widehat{G}J) \cong \operatorname{Ran}_{\widehat{G}}(\widehat{G} \circ \operatorname{Ran}_J J) \cong \operatorname{Ran}_{\widehat{G}} \widehat{G}$$

by Lemma 2.2.3(iv) and the fact that J is codense.

We assume without loss of generality that $\mathfrak{D}$ contains all the singleton diagrams in $\mathcal{A}$. Any $l \in \mathfrak{D}[\mathcal{A}]$ is $\operatorname{colim} y^{\mathrm{op}} D^{\mathrm{op}}$ in $[\mathcal{A}, \mathsf{Set}]$ for some $D\colon \mathcal{I} \to \mathcal{A} \in \mathfrak{D}$ and $y\colon \mathcal{A} \to [\mathcal{A}, \mathsf{Set}]^{\mathrm{op}}$ the Yoneda embedding. It follows from the continuity of a limit in its weight that the limit of $F\colon \mathcal{A} \to \mathcal{C}$ weighted by $\mathfrak{D}[\mathcal{A}](l, J-) = l$ is exactly $\lim FD$ when either side exists. Hence,

$$\begin{aligned}
\widehat{G}(\operatorname{Ran}_J J(l)) &= \widehat{G}\Big(\{\mathfrak{D}[\mathcal{A}](l, J-), J-\}\Big) \\
&\cong \widehat{G}(\lim JD) \\
&\cong \lim \widehat{G}JD && \text{by definition of } \widehat{G} \\
&\cong \{\mathfrak{D}[\mathcal{A}](l, J-), \widehat{G}J-\} \\
&= \operatorname{Ran}_J \widehat{G}J(l).
\end{aligned}$$

□

In fact, the argument used in the first paragraph of the previous proof shows that if

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "G"] \arrow[dr, "K"'] & & \mathcal{B}. \\
& \mathcal{D} \arrow[ur, "H"'] &
\end{tikzcd}$$

commutes, K is codense and H preserves $\operatorname{Ran}_K K$, then G and H have the same codensity monad. We can also obtain the analogous result for completions under a class of limit shapes, which is of course a special case of the previous proposition.

Corollary 4.3.18. *Let $G\colon \mathcal{A} \to \mathcal{B}$ be a functor with a pointwise codensity monad and $\mathfrak{C}$ be a class of categories such that $\mathcal{B}$ is $\mathfrak{C}$-complete. Let $\widehat{G}\colon \mathfrak{C}(\mathcal{A}) \to \mathcal{B}$ be the unique $\mathfrak{C}$-continuous extension of G. Then G and $\widehat{G}$ have the same codensity monad.*

Let $\mathfrak{C}$ be a class of small categories. Recall that $\mathfrak{C}(-)$ is a (colax-idempotent) pseudomonad on CAT. In [61], Tholen used this pseudomonad to generalise the concept of right adjoint and monadic functor. Explicitly, a functor $G\colon \mathcal{A} \to \mathcal{B}$ is a **right $\mathfrak{C}$-pro-adjoint** or **$\mathfrak{C}$-pro-monadic** if $\mathfrak{C}(G)\colon \mathfrak{C}(\mathcal{A}) \to \mathfrak{C}(\mathcal{B})$ is a right adjoint or monadic, respectively. Note that since $\mathfrak{C}(-)$ is a pseudofunctor, any right adjoint is a $\mathfrak{C}$-pro-adjoint.

Taking $\mathfrak{C}$ to be the class Prod of all sets (seen as small discrete categories), recovers Diers's notions of **right multiadjoint** and **multimonadic functor** defined in [23, p. 58] and [21, §3], respectively. The category $\mathsf{Prod}(\mathcal{A})$ is the free product completion of $\mathcal{A}$.

Lemma 4.3.19 (Tholen)**.** *A functor $G\colon \mathcal{A} \to \mathcal{B}$ is a right multiadjoint iff, for each $b \in \mathcal{B}$, the category $b{\downarrow}G$ has a set of connected components and each component has an initial object.*

Proof. The second condition is the original definition of right multiadjoint given by Diers [23, p. 58]. The equivalence with the definition here was proved by Tholen [61, Thm. 2.4]. ☐

That our definition of multimonadic functor coincides with that of Diers's is also due to Tholen [61, Thm. 4.1].

Our main reason to focus on the special case $\mathfrak{C} = \mathsf{Prod}$ is that Diers developed the corresponding theory quite extensively in [21, 23, 22]. For example, there are analogues of the monadicity theorems for multimonadic functors (see [21, §3 and §4]). There is also a sufficient condition for multimonadicity that is easy to check in practice.

Definition 4.3.20. A functor $G\colon \mathcal{A} \to \mathcal{B}$ is **relatively fully faithful** if every morphism of $\mathcal{A}$ is G-cartesian, i.e. for any pair of morphisms $f\colon X \to Z$ and $g\colon Y \to Z$ in $\mathcal{A}$ with the same codomain, and any $m\colon GX \to GY$ in $\mathcal{B}$ such that $Gg \cdot m = Gf$, there exists a unique $h\colon X \to Y$ such that $hg = f$ and $Gh = m$.

This condition may be summarised by the following diagram:

$$\begin{tikzcd} X \arrow[d, "\exists! h"'] \arrow[dr, "\forall f"] & & & GX \arrow[d, "\forall m"'] \arrow[dr, "Gf"] & \\ Y \arrow[r, "\forall g"'] & Z & {} \arrow[r, mapsto, "G"] & GY \arrow[r, "Gg"'] & GZ. \end{tikzcd}$$

It can also be understood as saying that the obvious functor $G/Z\colon \mathcal{A}/Z \to \mathcal{B}/GZ$ that G induces on the slice category is fully faithful for all $Z \in \mathcal{A}$.

The next proposition is part of Diers's [21, Prop. 6.0].

Proposition 4.3.21 (Diers)**.** *A functor that is both a right multiadjoint and relatively fully faithful is multimonadic.*

For ease of reference, we reproduce one last powerful result about multimonadic functors which can be found in Diers [22, p. 661]. It relates the multimonadicity of a functor to the monadicity of the induced product-preserving functor for categories over Set.

Theorem 4.3.22 (Diers)**.** *Let $G\colon \mathcal{A} \to \mathsf{Set}$ be a functor. Its unique product-preserving extension*

$$\widehat{G}\colon \mathsf{Prod}(\mathcal{A}) \longrightarrow \mathsf{Set}$$

is monadic iff G is multimonadic and satisfies:

$$\text{for any set } I,\ A \in \mathcal{A} \text{ and family of morphisms } (f_i\colon A \to A_i)_{i\in I} \text{ in } \mathcal{A}, \text{ if } (GA, Gf_i)_{i\in I} \text{ is a product cone, then } I \text{ has exactly one element.} \tag{D}$$

If this is the case, the corresponding monad is the codensity monad of G.

Condition (D) implies that G does not create any products indexed by a set I, unless I is a singleton. If G reflects products (e.g. if G is fully faithful), then these two conditions are equivalent.

Examples 4.3.23.

(i) The argument in Example 4.3.4(iv) shows, in the light of Lemma 4.3.19, that the forgetful functor $V\colon \mathsf{Field} \hookrightarrow \mathsf{CRing}$ is a right multiadjoint. It is clear from the definition that right multiadjoints are closed under composition, so the composite

$$\mathsf{Field} \xhookrightarrow{V} \mathsf{CRing} \xrightarrow{U} \mathsf{Set}$$

of forgetful functors is also a right multiadjoint. Since all field homomorphisms are injective, it is easy to see that UV is relatively fully faithful, and hence multimonadic by Proposition 4.3.21. Moreover, UV reflects limits, because both U and V do, and it is clear that it does not create any nontrivial products. It follows from Theorem 4.3.22 that the functor $\widehat{UV}\colon \mathsf{Prod}(\mathsf{Field}) \to \mathsf{Set}$ taking a formal product of fields to the product of their underlying sets is monadic, and that it is the monadic reflection of UV.

Since CRing has all products and U preserves them, the functor $\widehat{UV}$ factors as

$$\mathsf{Prod}(\mathsf{Field}) \xrightarrow{\widehat{V}} \mathsf{CRing} \xrightarrow{U} \mathsf{Set}$$

where U is also monadic. It follows that $\widehat{H}$ is an algebraic functor over Set and hence monadic by Theorem 2.4.5. By Corollary 4.3.18, it induces the codensity monad of V. In other words, $\mathsf{Prod}(\mathsf{Field}) \to \mathsf{CRing}$ is the monadic reflection of $\mathsf{Field} \hookrightarrow \mathsf{CRing}$.

(ii) Similarly, Example 4.3.4(v) shows that $\mathsf{E}_E = (-) + E\colon \mathsf{Set} \to \mathsf{Set}$ is a right multiadjoint. In fact, it is also relatively fully faithful. Indeed, let

$f\colon X \to Z$ and $g\colon Y \to Z$ be functions, and $m\colon X + E \to Y + E$ be such that $(g + E) \cdot m = f + E$. It follows that m must act as the identity on E, so it is of the form $h + E$ for a unique $h\colon X \to Y$ such that $gh = f$.

The functor E_E satisfies (D) iff E has at least two elements. It is clear that it fails for $E = \varnothing$ and $E = 1$ (the latter one by taking $I = \varnothing$). Now let $e, e' \in E$ be distinct, I and X be sets, and $(f_i\colon X \to X_i)_{i \in I}$ be a cone such that $(X + E, f_i + X)_{i \in I}$ is a product cone. Since $X + E$ is not terminal, I is nonempty. If I has at least two elements, i and i', then there is some element of $X + E$ that maps to e under $f_i + E$ and to e' under $f_{i'} + E$, but this is impossible. It follows that I is a singleton.

Theorem 4.3.22 then gives a proper class of monadic functors $\widehat{\mathsf{E}_E}\colon \mathsf{Prod}(\mathsf{Set}) \to \mathsf{Set}$ (one for each E with at least two elements), sending the formal product of $(X_i)_{i \in I}$ to the cartesian product $\prod_{i \in I}(X_i + E)$.

Remark 4.3.24. Given that Corollary 4.3.18 applies to more general limit completions than those under products, it is possible to imagine generalisations of Theorem 4.3.22 to $\mathfrak{C}$-pro-monadic functors, where $\mathfrak{C}$ is perhaps a sound doctrine of limits. We leave this investigation for future work.

Chapter 5

Distributive laws and pushforwards

When first learning basic arithmetic, one encounters the familiar equations

$$x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3 \qquad \text{and} \qquad (x_1 + x_2) \cdot x_3 = x_1 \cdot x_3 + x_2 \cdot x_3.$$

They are commonly referred to as the distributive law of multiplication over addition. From the point of view of algebraic theories, or more generally monads, these equations allow us to rewrite any product of sums as a sum of products. Writing A and M for the free abelian group and the free monoid monads on Set, this gives a natural transformation $\delta\colon \mathsf{MA} \to \mathsf{AM}$ which is compatible with the unit and multiplication of either monad. This natural transformation can be used to endow the composite AM with a monad structure, which is exactly the free (associative, unital) ring monad.

Beck realised that there is nothing special about the monads A and M in this situation: such a natural transformation can always be used to endow the composite endofunctor with a monad structure. He called this structure a **distributive law** (Definition 2.3.11), as a nod to the quintessential example above. Recall from Theorems 2.3.13 and 2.3.14, which are essentially due to Beck, that if T and S are monads on a category $\mathcal{B}$ then there is a one-to-one correspondence between

(i) distributive laws of T over S;

(ii) monad structures on ST such that $S\eta^T\colon S \to ST$ and $\eta^S T\colon T \to ST$ are monad maps and the middle unitary law holds;

(iii) monad lifts $\overline{S}$ of S along the monadic functor $U^T\colon \mathcal{B}^T \to \mathcal{B}$.

Such a monad lift amounts to a lax monad functor $(U^T, 1)\colon (\mathcal{B}^T, \overline{S}) \to (\mathcal{B}, S)$. Note that the passage from (iii) to (ii) is given by taking the pushforward $U^T_\sharp \overline{S}$, while $U^T_\sharp 1 = T$.

Wolff [63] studied what happens to this equivalence if one relaxes (iii) above to a monad lift $\overline{S}$ along a mere right adjoint $U\colon \mathcal{A} \to \mathcal{B}$ which induces the monad T. The passage from (iii) to (i), or equivalently (ii), remains valid. Remarkably, his Theorem 1.3 gives a converse statement under some mild assumptions: given a distributive law of T over S, one can construct a lift of S along U, initial among those inducing the given distributive law, as soon as $\mathcal{A}$ has and U preserves certain colimits.

In this chapter, we take this one step further. Codensity monads give a way to replace an arbitrary functor by a monadic one; what happens if we further relax (iii) above to a monad lift $\overline{S}$ of S along an arbitrary functor $G\colon \mathcal{A} \to \mathcal{B}$ whose codensity monad is T? It turns out that the resulting pushforward $G_\sharp\overline{S}$ behaves in many ways like the composite monad in (ii). In particular, there are monad maps $S \to G_\sharp\overline{S}$ and $T \to G_\sharp\overline{S}$ satisfying an analogue of the middle unitary law (Lemma 5.1.1). Of course, the monad $G_\sharp\overline{S}$ can only correspond to the composite induced by a distributive law of T over S if $G_\sharp\overline{S}$ actually coincides with ST as an endofunctor. This is quite a strong condition, so we also consider the corresponding question for the monad induced by the more general concept of a weak distributive law.

As in Chapter 4, we work in the generality of monads in a 2-category $\mathcal{K}$. In Section 5.1 we tackle the case of classical distributive laws. We show that if $g\colon x \to y$ is a 1-cell and s is a monad on y whose underlying 1-cell preserves $\mathrm{ran}_g\, g$, then any lift of s along g gives rise to a distributive law of $g_\sharp 1$ over s whose induced composite monad is exactly $g_\sharp\overline{s}$ (Theorem 5.1.5). In fact, if the counit of $\mathrm{ran}_g\, g$ is invertible (as is the case e.g. when g is a fully faithful functor in CAT), then there is a unique distributive law of this type, as soon as s admits a lift along g. This allows us to prove that there is a unique distributive law between certain pairs of monads (Example 5.1.7).

The remaining sections study the analogous situation for a weakening of distributive laws introduced by Böhm [14]. Section 5.2 introduces the necessary background to understand these weak distributive laws and appreciate their importance. In particular, we will focus on a variant of weak distributive laws which we call π-weak (Definition 5.2.3). Every π-weak distributive law of a monad t over a monad s induces a weak composite monad, i.e. a compatible monad structure on a retract of the composite st. Section 5.3 is devoted to proving theorems characterising (π-)weak distributive laws in terms of their weak composites in the style of Beck (Theorems 5.3.2 and 5.3.4). Section 5.4 then uses this characterisation to relate π-weak distributive laws to pushforwards of lifted monads as was done in Section 5.1 for classical distributive laws (Theorem 5.4.4). This allows use to recover Garner's [28] weak distributive law of the ultrafilter monad over the powerset monad from the perspective of pushforward monads (Example 5.4.5).

There is at least one other way in which distributive laws and pushforward monads may interact which is not pursued in this chapter. As we saw at the end of Section 2.3, distributive laws in a 2-category $\mathcal{K}$ are exactly monads in $\mathsf{MND}(\mathcal{K})$. As such, one can in theory speak of pushing a distributive law forward along a lax monad functor. The author explored this possibility but was faced with a crucial obstacle: there seems to be no simple way of computing right extensions in $\mathsf{MND}(\mathcal{K})$, even when $\mathcal{K} = \mathsf{CAT}$. A promising approach is to consider the locally fully faithful 2-functor $\mathsf{MND}(\mathsf{CAT}) \hookrightarrow \mathsf{CAT}^{\to}$ of Theorem 2.3.7, which reflects extensions by Lemma 2.2.3(i). Restricting to small categories, we have that $\mathsf{Cat}^{\to}$ is isomorphic to the 2-category $\mathsf{Cat}(\mathsf{Set}^{\to})$ of categories internal to the presheaf topos $\mathsf{Set}^{\to}$, because category objects are defined in terms of limits and these are computed objectwise in $\mathsf{Set}^{\to}$. However, even in this setting we have not been able to obtain a simple ‘pointwise’ formula for right extensions in $\mathsf{Cat}(\mathsf{Set}^{\to})$.

5.1 Pushforwards and distributive laws

For the remainder of this section, let $g\colon x \to y$ be a 1-cell in a 2-category $\mathcal{K}$ and let $(g,\theta)\colon (x,\bar{s}) \to (y,s)$ be a lax monad functor. We assume that $g_\sharp 1$ and $g_\sharp \bar{s}$ exist, and write ϵ^x and $\epsilon^{\bar{s}}$ for their respective counits. Already in this minimal scenario two monad maps are available:

- $\psi\colon s \to g_\sharp \bar{s}$, corresponding under Theorem 4.1.5 to the lax monad functor (g,θ). It is the unique 2-cell such that
$$\epsilon^{\bar{s}} \cdot \psi g = \theta; \qquad (\psi)$$
- $\varphi\colon g_\sharp 1 \to g_\sharp \bar{s}$, the image of the monad map $\eta^{\bar{s}}\colon 1 \to \bar{s}$ under the (partial) functor $g_\sharp$ (see Lemma 4.1.4 and Example 2.3.4). It is the unique 2-cell such that
$$\epsilon^{\bar{s}} \cdot \varphi g = \eta^{\bar{s}} \cdot \epsilon^x. \qquad (\varphi)$$

Moreover, these maps satisfy a version of the middle unitary law of Theorem 2.3.13, as the next lemma shows. In the ordinary middle unitary law, ν below is just an identity. In this generality, we need to involve a mediating 2-cell.

Lemma 5.1.1. *The diagram*

$$\begin{tikzcd}
 & s(g_\sharp 1) \arrow[dl, "\psi\varphi"'] \arrow[dr, "\nu"] & \\
(g_\sharp \bar{s})^2 \arrow[rr, "\mu^{g_\sharp \bar{s}}"'] & & g_\sharp \bar{s}
\end{tikzcd}$$

commutes, where ν is the unique 2-cell such that

$$\epsilon^{\bar{s}} \cdot \nu g = \theta \cdot s\epsilon^x. \qquad (\nu)$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
s(g_\sharp 1)g \arrow[rr, "s\epsilon^x"] \arrow[dd, "s\varphi g"'] & & sg \arrow[rr, "\theta"] \arrow[dd, "sg\eta^{\bar{s}}"'] & & g\bar{s} \arrow[dd, "g\bar{s}\eta^{\bar{s}}"] \\
 & (\varphi) & & (*) & \\
s(g_\sharp \bar{s})g \arrow[rr, "s\epsilon^{\bar{s}}"'] \arrow[dd, "\psi(g_\sharp \bar{s})g"'] & & sg\bar{s} \arrow[rr, "\theta\bar{s}"'] \arrow[dd, "\psi g\bar{s}"'] & & g\bar{s}^2 \arrow[dd, equal] \\
 & (*) & & (\psi) & \\
(g_\sharp \bar{s})^2 g \arrow[rr, "(g_\sharp \bar{s})\epsilon^{\bar{s}}"'] \arrow[dd, "\mu^{g_\sharp \bar{s}}g"'] & & (g_\sharp \bar{s})g\bar{s} \arrow[rr, "\epsilon^{\bar{s}}\bar{s}"'] & & g\bar{s}^2 \arrow[dd, "g\mu^{\bar{s}}"] \\
 & & (\epsilon^{\bar{s}}.\mu) & & \\
(g_\sharp \bar{s})g \arrow[rrrr, "\epsilon^{\bar{s}}"'] & & & & g\bar{s}.
\end{tikzcd}$$

Altogether, this shows that

$$\epsilon^{\overline{s}} \cdot \left(\mu^{g_\sharp \overline{s}} \cdot \psi\varphi\right) g = g\mu^{\overline{s}} \cdot g\overline{s}\eta^{\overline{s}} \cdot \theta \cdot s\epsilon^{x} \overset{(\overline{s}.r)}{=} \theta \cdot s\epsilon^{x}.$$

The result follows by the uniqueness of ν. $\square$

In the absence of a distributive law of $g_\sharp 1$ over s, one cannot ask for ν to be a monad map, because its domain may not be a monad. If we do have a distributive law $\delta\colon (g_\sharp 1)s \to s(g_\sharp 1)$, then ν will be a monad map as soon as δ is compatible with θ in the sense of the following proposition.

Proposition 5.1.2. *Let $\delta\colon (g_\sharp 1)s \to s(g_\sharp 1)$ be a distributive law. If*

$$\begin{tikzcd}
(g_\sharp 1)sg \arrow[rr, "\delta g"] \arrow[d, "(g_\sharp 1)\theta"'] & & s(g_\sharp 1)g \arrow[d, "s\epsilon^{x}"] \\
(g_\sharp 1)g\overline{s} \arrow[r, "\epsilon^{x}\overline{s}"'] & g\overline{s} & sg \arrow[l, "\theta"]
\end{tikzcd} \tag{5.1}$$

commutes, then $\nu\colon s \circ_\delta g_\sharp 1 \to g_\sharp\overline{s}$ in Lemma 5.1.1 is a monad map.

Moreover, if $\overline{s}$ is a genuine lift of s and $g_\sharp 1$ is a genuine extension of g, i.e. if θ and ϵ^{x} are invertible, respectively, then (5.1) commutes.

Proof. By Theorem 4.1.5, ν is a monad map iff $(g, \theta \cdot s\epsilon^{x})$ is a lax monad functor $(x, \overline{s}) \to (y, s \circ_\delta g_\sharp 1)$. We check the axioms for the latter.

First, note that

$$\begin{tikzcd}
g \arrow[r, "\eta^{s} g"] \arrow[dd, "g\eta^{\overline{s}}"'] \arrow[rdd, phantom, "(\theta.\eta)"] & sg \arrow[rd, "s\eta^{g_\sharp 1} g"] \arrow[dd, equal] \arrow[dd, phantom, shift left=6ex, "(\epsilon^{x}.\eta)"] & \\
& & s(g_\sharp 1)g \arrow[ld, "s\epsilon^{x}"] \\
g\overline{s} & sg \arrow[l, "\theta"] &
\end{tikzcd}$$

commutes. Hence,

$$(\theta \cdot s\epsilon^{x}) \cdot \eta^{s}\eta^{g_\sharp 1} g = g\eta^{\overline{s}},$$

so $(g, \theta \cdot s\epsilon^{x})$ is compatible with the unit.

Second, consider the commutative diagram

$$\begin{tikzcd}
s(g_\sharp 1)s(g_\sharp 1)g \arrow[r, "s(g_\sharp 1)s\epsilon^{x}"] \arrow[d, "s\delta(g_\sharp 1)g"'] \arrow[rd, phantom, "(*)"] & s(g_\sharp 1)sg \arrow[r, "s(g_\sharp 1)\theta"] \arrow[d, "s\delta g"'] & s(g_\sharp 1)g\overline{s} \arrow[dd, "s\epsilon^{x}\overline{s}"] & \\
s^{2}(g_\sharp 1)^{2}g \arrow[r, "s^{2}(g_\sharp 1)\epsilon^{x}"] \arrow[d, "s^{2}\mu^{g_\sharp 1}g"'] \arrow[rd, phantom, "(\epsilon^{x}.\mu)"] & s^{2}(g_\sharp 1)g \arrow[d, "s^{2}\epsilon^{x}"'] \arrow[r, phantom, "(5.1)"] & & \\
s^{2}(g_\sharp 1)g \arrow[r, "s^{2}\epsilon^{x}"] \arrow[d, "\mu^{s}(g_\sharp 1)g"'] \arrow[rd, phantom, "(*)"] & s^{2}g \arrow[r, "s\theta"] \arrow[d, "\mu^{s}g"'] \arrow[rrd, phantom, "(\theta.\mu)"] & sg\overline{s} \arrow[r, "\theta\overline{s}"] & g\overline{s}^{2} \arrow[d, "g\mu^{\overline{s}}"] \\
s(g_\sharp 1)g \arrow[r, "s\epsilon^{x}"'] & sg \arrow[rr, "\theta"'] & & g\overline{s}.
\end{tikzcd}$$

This shows that

$$(\theta \cdot s\epsilon^x) \cdot \left\{\mu^s \mu^{g_\sharp 1} \cdot s\delta(g_\sharp 1)\right\} g = g\mu^{\bar{s}} \cdot (\theta \cdot s\epsilon^x)\bar{s} \cdot s(g_\sharp 1)(\theta \cdot s\epsilon^x),$$

so $(g, \theta \cdot s\epsilon^x)$ is compatible with the multiplication.

For the last part, we will actually prove a stronger statement: if $\eta^{g_\sharp 1} sg$ is epic, then (5.1) commutes. The result follows because, if ϵ^x is invertible, then so is $\eta^{g_\sharp 1} g = \epsilon^{-1}$ and further, if θ is invertible, the so is $\eta^{g_\sharp 1} sg = (g_\sharp 1)\theta^{-1} \cdot \eta^{g_\sharp 1} g\bar{s} \cdot \theta$. Now consider the diagram

$$\begin{tikzcd}
g\bar{s} \arrow[d, "\eta^{g_\sharp 1}g\bar{s}"'] \arrow[dd, equal, bend right=60, "(\epsilon^x.\eta)"] \arrow[dr, phantom, "(*)"] & sg \arrow[l, "\theta"'] \arrow[d, two heads, "\eta^{g_\sharp 1}sg"'] \arrow[r, equal] \arrow[dr, phantom, "(\delta.\eta^{g_\sharp 1})"] & sg \arrow[d, "s\eta^{g_\sharp 1}g"] \arrow[dd, equal, bend left=60, "(\epsilon^x.\eta)"'] \\
(g_\sharp 1)g\bar{s} \arrow[d, "\epsilon^x\bar{s}"] & (g_\sharp 1)sg \arrow[l, "(g_\sharp 1)\theta"] \arrow[r, "\delta g"'] & s(g_\sharp 1)g \arrow[d, "s\epsilon^x"'] \\
g\bar{s} & & sg. \arrow[ll, "\theta"]
\end{tikzcd}$$

The outside face commutes trivially. If $\eta^{g_\sharp 1} sg$ is epic, then the bottom pentagon, i.e. (5.1), commutes. ☐

The next proposition can be thought of as a partial converse to the previous one. We will first need a simple lemma about monad maps.

Lemma 5.1.3. *Let r, s and t be monads on x, and $\alpha\colon r \to s$ and $\beta\colon s \to t$ be 2-cells with β monic. If β and $\beta \cdot \alpha$ are monad maps, then so is α.*

Proof. Since β is monic,

$$\beta \cdot \alpha \cdot \eta^r \overset{((\beta\cdot\alpha).\eta)}{=} \eta^t \overset{(\beta.\eta)}{=} \beta \cdot \eta^s,$$

implies that $\alpha \cdot \eta^r = \eta^s$. Similarly, α is compatible with the multiplication. ☐

If $\alpha\colon s \to t$ is a monic 2-cell and t has a monad structure, then s admits at most one monad structure which makes α a monad map. This happens precisely when $\mu^t \cdot \alpha\alpha$ and η^t factor through α. In this situation, we say that the **monad structure of t restricts along** α.

Proposition 5.1.4. *Let $\nu\colon s(g_\sharp 1) \to g_\sharp \bar{s}$ be as in Lemma 5.1.1. If ν is monic and the monad structure of $g_\sharp \bar{s}$ restricts along it, then the restricted monad structure is the composite monad given by a distributive law of $g_\sharp 1$ over s satisfying (5.1). Moreover, this is the unique such distributive law.*

Proof. Let η and μ be the unit and multiplication of the restricted monad structure on $s(g_\sharp 1)$. It is easy to see that

$$\nu \cdot s\eta^{g_\sharp 1} = \psi \qquad \text{and} \qquad \nu \cdot \eta^s(g_\sharp 1) = \varphi, \tag{5.2}$$

since

$$\begin{tikzcd}
 & s(g_\sharp 1)g \arrow[r, "\nu g"] \arrow[d, "s\epsilon^x"] \arrow[dr, phantom, "(\nu)"] & (g_\sharp \overline{s})g \arrow[d, "\epsilon^{\overline{s}}"] \\
sg \arrow[ur, bend left, "s\eta^{g_\sharp 1}g"] \arrow[r, equal] \arrow[ur, phantom, "(\eta^{g_\sharp 1})"] & sg \arrow[r, "\theta"'] & g\overline{s},
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(g_\sharp 1)g \arrow[r, "\eta^s(g_\sharp 1)g"] \arrow[d, "\epsilon^x"'] \arrow[dr, phantom, "(*)"] & s(g_\sharp 1)g \arrow[r, "\nu g"] \arrow[d, "s\epsilon^x"'] \arrow[dr, phantom, "(\nu)"] & (g_\sharp \overline{s})g \arrow[d, "\epsilon^{\overline{s}}"] \\
g \arrow[r, "\eta^s g"'] \arrow[rr, bend right, "g\eta^{\overline{s}}"'] \arrow[rr, phantom, bend right=20, "(\theta.\eta)"] & sg \arrow[r, "\theta"'] & g\overline{s}.
\end{tikzcd}$$

Since ν, ψ and φ are monad maps and ν is monic, it follows from Lemma 5.1.3 that $s\eta^{g_\sharp 1}$ and $\eta^s(g_\sharp 1)$ are monad maps. In particular, since they are compatible with the unit, $\eta = \eta^s\eta^{g_\sharp 1}$.

Now consider the diagram

$$\begin{tikzcd}
s(g_\sharp 1) \arrow[r, "s\eta^{g_\sharp 1}\eta^s(g_\sharp 1)"] \arrow[dr, bend right, "\psi\varphi"'] \arrow[dr, phantom, "(5.2)"] & \{s(g_\sharp 1)\}^2 \arrow[r, "\mu"] \arrow[d, "\nu\nu"'] \arrow[dr, phantom, "(\nu.\mu)"] & s(g_\sharp 1) \arrow[d, hook, "\nu"] \\
 & (g_\sharp \overline{s})^2 \arrow[r, "\mu^{g_\sharp \overline{s}}"'] & g_\sharp \overline{s}
\end{tikzcd}$$

By Lemma 5.1.1, the bottom composite is ν. Since ν is monic, we conclude that $\mu \cdot s\eta^{g_\sharp 1}\eta^s(g_\sharp 1) = 1_{s(g_\sharp 1)}$. By Theorem 2.3.13, this shows that the monad structure of $s(g_\sharp 1)$ comes from a distributive law of $g_\sharp 1$ over s.

That the corresponding distributive law $\delta\colon (g_\sharp 1)s \to s(g_\sharp 1)$ satisfies (5.1) follows from the commutativity of

$$\begin{tikzcd}
(g_\sharp 1)sg \arrow[r, "\eta^s(g_\sharp 1)s\eta^{g_\sharp 1}g"] \arrow[d, "(g_\sharp 1)\psi g"] \arrow[dd, bend right=60, "(g_\sharp 1)\theta"'] \arrow[dr, phantom, "(5.2)"] & s(g_\sharp 1)s(g_\sharp 1)g \arrow[r, "\mu g"] \arrow[d, "\nu\nu g"'] \arrow[dr, phantom, "(\nu.\mu)"] & s(g_\sharp 1)g \arrow[d, "\nu g"] \arrow[dr, bend left, "s\epsilon^x"] & \\
(\psi)\ (g_\sharp 1)(g_\sharp \overline{s})g \arrow[r, "\varphi(g_\sharp \overline{s})g"'] \arrow[d, "(g_\sharp 1)\epsilon^{\overline{s}}"] \arrow[dr, phantom, "(*)"] & (g_\sharp \overline{s})^2 g \arrow[r, "\mu^{g_\sharp \overline{s}}g"'] \arrow[d, "(g_\sharp \overline{s})\epsilon^{\overline{s}}"'] & (g_\sharp \overline{s})g \arrow[dd, "\epsilon^{\overline{s}}"'] \arrow[r, phantom, "(\nu)"] & sg \arrow[ddl, bend left, "\theta"] \\
(g_\sharp 1)g\overline{s} \arrow[r, "\varphi g\overline{s}"'] \arrow[d, "\epsilon^x\overline{s}"'] \arrow[dr, phantom, "(\varphi)"] & (g_\sharp \overline{s})g\overline{s} \arrow[d, "\epsilon^{\overline{s}}\overline{s}"'] \arrow[r, phantom, "(\epsilon^{\overline{s}}.\mu)"] & & \\
g\overline{s} \arrow[r, "g\eta^{\overline{s}}\overline{s}"'] \arrow[rr, bend right, equal] \arrow[rr, phantom, bend right=15, "(\overline{s}.l)"] & g\overline{s}^2 \arrow[r, "g\mu^{\overline{s}}"'] & g\overline{s}. &
\end{tikzcd}$$

Lastly, suppose $\delta'\colon (g_\sharp 1)s \to s(g_\sharp 1)$ is a distributive law satisfying (5.1). By Proposition 5.1.2, ν is a monad map. Since ν is monic, this means that the monad structure $s \circ_{\delta'} g_\sharp x$ is precisely the restriction along ν of that of $g_\sharp \overline{s}$. Thus, δ' and δ induce the same composite monad, which by Theorem 2.3.13 implies that $\delta' = \delta$. □

We are now ready for the main result of this section, which generalises the construction of a distributive law from a monad lift along a monadic functor. We call a lax monad functors (g, θ) **strong** if θ is invertible.

Theorem 5.1.5. *Let $g\colon x \to y$ be a 1-cell in $\mathcal{K}$ and s be a monad on y whose 1-cell part preserves $\operatorname{ran}_g g$. Every strong monad functor $(g, \theta)\colon (x, \bar{s}) \to (y, s)$ induces a distributive law δ of $g_\sharp 1$ over s satisfying (5.1) such that the resulting composite monad is $g_\sharp \bar{s}$.*

Moreover, if the counit ϵ^x of $\operatorname{ran}_g g$ is invertible and there is at least one strong monad functor $(g, \theta)\colon (x, \bar{s}) \to (y, s)$, then there exists a unique distributive law of $g_\sharp 1$ over s, and the monad isomorphism class of $g_\sharp \bar{s}$ is independent of the choice of lift $\bar{s}$.

Proof. Since s preserves $\operatorname{ran}_g g$, the 1-cell $s(g_\sharp 1)$ is a right extension of sg along g with counit $s\epsilon^x$. Since $\theta\colon sg \to g\bar{s}$ is an isomorphism, $g_\sharp \bar{s}$ is given by $s(g_\sharp 1)$ with counit $\theta \cdot s\epsilon^x$. It follows that the 2-cell $\nu\colon s(g_\sharp 1) \to g_\sharp \bar{s}$ of Lemma 5.1.1 is the identity, and hence the monad structure of $g_\sharp \bar{s}$ trivially restricts along it. By Proposition 5.1.4, $g_\sharp \bar{s} = s(g_\sharp 1)$ is the monad induced by the unique distributive law of $g_\sharp 1$ over s satisfying (5.1).

Now suppose ϵ^x is invertible and let $(g, \theta)\colon (x, \bar{s}) \to (y, s)$ be a strong monad functor. By Proposition 5.1.2, any distributive law δ of $g_\sharp 1$ over s satisfies (5.1) and hence is equal to the one above. Any other strong monad functor $(g, \theta')\colon (x, \bar{s}') \to (y, s)$ induces this unique distributive law, so that $g_\sharp \bar{s}'$ and $g_\sharp \bar{s}$ are isomorphic monads. ☐

Example 5.1.6. If g is a right adjoint, then s preserves $\operatorname{ran}_g g$, and so every strong monad functor $(g, \theta)\colon (x, \bar{s}) \to (y, s)$ induces a distributive law of $g_\sharp 1$ over s. This recovers the main statement of Lemma 1.1 of Wolff [63]. If $g = u^t$ for some monad t on y, this is the familiar construction of a distributive law from a monad lifted along u^t. In particular, if $g = u^t$ and $(u^t, \theta)\colon (y^t, s^\delta) \to (y, s)$ is the monad lift induced by a distributive law δ of t over s, then (5.1) always commutes.

Example 5.1.7. As we will see in Section 6.1.1, Theorem 5.1.5 implies that there is a unique distributive law of the ultrafilter monad on Set over

(i) the exception monad $- + E$, where E is a finite set;

(ii) the free M-set monad, where M is a finite monoid;

(iii) the reader monad $(-)^I$, where I is a finite set;

(iv) the state monad $(S \times -)^S$, where I is a finite set.

The machinery needed to prove Theorem 5.1.5 can also be used to prove that no distributive law of a given type exist.

Proposition 5.1.8. *Let $(g, \theta)\colon (x, \bar{s}) \to (y, s)$ be a strong monad functor and ϵ^x be invertible. Suppose the 2-cell $\nu\colon s(g_\sharp 1) \to g_\sharp \bar{s}$ in Lemma 5.1.1 is monic. Then there is a (unique) distributive law of $g_\sharp 1$ over s iff the monad structure of $g_\sharp \bar{s}$ restricts along ν.*

Proof. By Proposition 5.1.2, any distributive law satisfies (5.1) for any strong monad functor (g, θ) as in the statement. The backwards implication is then given by Proposition 5.1.4. For the forward one, Proposition 5.1.2 shows that ν is a monad map. Since it is monic, this is precisely what it means for the monad structure of $g_{\sharp}\bar{s}$ to restrict along ν. ☐

Proposition 6.1.19 below uses this characterisation to show that there is no distributive law of the ultrafilter monad over the finite powerset monad.

5.2 The weak theory of monads

In [14], Böhm develops a generalisation of the formal theory of monads to allow lax monad functors which are not necessarily compatible with the unit. The result is a 2-category $\mathsf{EM}^w(\mathcal{K})$ where the 0-cells are monads in $\mathcal{K}$ and the 1-cells $(x, t) \to (y, s)$ are pairs (g, φ) of a 1-cell $g \colon x \to y$ and a 2-cell $\varphi \colon sg \to gt$ such that $(\varphi.\mu)$ commutes. The price of relaxing the requirement on the 1-cells is paid by imposing an extra condition on the 2-cells, when compared to the 2-cells of $\mathsf{EM}(\mathcal{K})$, which is needed to make $\mathsf{EM}^w(\mathcal{K})$ a 2-category. Explicitly, a 2-cell $(g, \varphi) \to (h, \psi)$ in $\mathsf{EM}^w(\mathcal{K})$ is a 2-cell $\alpha \colon g \to ht$ in $\mathcal{K}$ such that

$$\begin{tikzcd}
sg \arrow[r, "\varphi"] \arrow[d, "s\alpha"'] & gt \arrow[r, "\alpha t"] & ht^2 \arrow[d, "h\mu^t"] \\
sht \arrow[r, "\psi t"'] & ht^2 \arrow[r, "h\mu^t"'] & ht
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
g \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & ht \arrow[r, "\eta^s ht"] & sht \arrow[d, "\psi t"] \\
ht & & ht^2 \arrow[ll, "h\mu^t"]
\end{tikzcd}$$

commute.

Just as $\mathsf{EM}(\mathcal{K})$ contains $\mathsf{Mnd}(\mathcal{K})$ as a 1-full wide sub-2-category, Böhm defines, not one, but two 1-full wide sub-2-categories of $\mathsf{EM}^w(\mathcal{K})$ which approximate the role of $\mathsf{Mnd}(\mathcal{K})$ in $\mathsf{EM}(\mathcal{K})$. These are the 2-categories $\mathsf{Mnd}^\iota(\mathcal{K})$ and $\mathsf{Mnd}^\pi(\mathcal{K})$ of [14, Cor. 1.4]. Their intersection $\mathsf{Mnd}^\iota(\mathcal{K}) \cap \mathsf{Mnd}^\pi(\mathcal{K})$ is also a 1-full wide subcategory, which we denote by $\mathsf{Mnd}^w(\mathcal{K})$. The relationships between all of these 2-categories can be summarised by the diagram

$$\begin{tikzcd}
\mathsf{Mnd}(\mathcal{K}) \arrow[rr] \arrow[dd, hook] & & \mathsf{Mnd}^w(\mathcal{K}) \arrow[dl, hook] \arrow[dr, hook] & \\
& \mathsf{Mnd}^\iota(\mathcal{K}) \arrow[dr, hook] & & \mathsf{Mnd}^\pi(\mathcal{K}) \arrow[dl, hook] \\
\mathsf{EM}(\mathcal{K}) \arrow[rr] & & \mathsf{EM}^w(\mathcal{K}), &
\end{tikzcd}$$

where the hooked arrows are inclusions of 1-full sub-2-categories and the horizontal ones are inclusions of 2-full sub-2-categories, all of which are wide.

Our focus will be on $\mathsf{Mnd}^\pi(\mathcal{K})$ because it has the strongest connection to pushforwards of lifted monads. We formally introduce its data now. That these data actually form a 2-category is the content of Böhm [14, Cor. 1.4].

Definition 5.2.1 (Böhm)**.** Let $\mathsf{Mnd}^{\pi}(\mathcal{K})$ be the 2-category where

- 0-cells are monads in $\mathcal{K}$, just like in $\mathsf{Mnd}(\mathcal{K})$;
- 1-cells $(x,t) \to (y,s)$ are pairs (g,φ) such that $(\varphi.\mu)$ commutes;
- 2-cells $(g,\varphi) \to (h,\psi)$ are 2-cells $\alpha\colon g \to h$ such that the following diagram commutes:

$$\begin{tikzcd}
gt \arrow[d, "\alpha t"'] & sg \arrow[l, "\varphi"'] \arrow[rr, "s\alpha"] & & sh \arrow[d, "\psi"] \\
ht \arrow[r, "\eta^s ht"'] & sht \arrow[r, "\psi t"'] & ht^2 \arrow[r, "h\mu^t"'] & ht.
\end{tikzcd}$$

Composition and identities are as in $\mathsf{Mnd}(\mathcal{K})$.

Just as a monad in $\mathsf{Mnd}(\mathcal{K})$ is a distributive law and a monad in $\mathsf{EM}(\mathcal{K})$ is a **wreath**, we will call the data of a monad in $\mathsf{Mnd}^{w}(\mathcal{K})$ a weak distributive law (following Garner's terminology in [28, §3.2]) and that of a monad in $\mathsf{EM}^{w}(\mathcal{K})$ a **weak wreath**. In simple terms:

Definition 5.2.2. Let s and t be monads on x. A **weak distributive law** of t over s is a 2-cell $\delta\colon ts \to st$ such that $(\delta.\eta^s)$, $(\delta.\mu^s)$ and $(\delta.\mu^t)$ in Definition 2.3.11 commute.

We do not write down the explicit data that make a weak wreath for two reasons. First, it involves two endo-1-cells, but only one of them is required to be a monad. This is too general for our purposes of relating such structures to the pushforward of lifted monads, since in the latter case we always start with two monads. Second, Böhm [14, Thm. 2.3] shows that weak wreaths correspond bijectively to certain premonads in $\mathcal{K}$. One may take this correspondence, stated as Theorem 5.2.6 below, as an implicit definition of a weak wreath.

The structure that we will be most concerned with is that of a monad in $\mathsf{Mnd}^{\pi}(\mathcal{K})$, which we call a π-weak distributive law. In simple terms:

Definition 5.2.3. Let s and t be monads on x. A **π-weak distributive law** of t over s is a 2-cell $\delta\colon ts \to st$ such that $(\delta.\mu^t)$ and the following commute:

$$\begin{tikzcd}
t \arrow[rrr, "t\eta^s"] \arrow[d, "\eta^s t"'] & & & ts \arrow[d, "\delta"] \\
st \arrow[r, "\eta^t st"'] & tst \arrow[r, "\delta t"'] \arrow[u, phantom, "(\delta.\mathrm{w}\eta^s)"] & st^2 \arrow[r, "s\mu^t"'] & st,
\end{tikzcd}
\qquad
\begin{tikzcd}
s^2t \arrow[d, "\mu^s t"'] & sts \arrow[l, "s\delta"'] & ts^2 \arrow[l, "\delta s"'] \arrow[r, "t\mu^s"] & ts \arrow[d, "\delta"] \\
st \arrow[r, "\eta^t st"'] & tst \arrow[r, "\delta t"'] \arrow[u, phantom, "(\delta.\mathrm{w}\mu^s)"] & st^2 \arrow[r, "s\mu^t"'] & st.
\end{tikzcd}$$

Since $\mathsf{Mnd}^{w}(\mathcal{K}) \subseteq \mathsf{Mnd}^{\pi}(\mathcal{K}) \subseteq \mathsf{EM}^{w}(\mathcal{K})$, every weak distributive law is a π-weak distributive law, and every π-weak distributive law gives rise to a weak wreath.

Given a monad t on x and an endo-1-cell s of x, a wreath $\delta\colon ts \to st$ can be used to equip st with a monad structure. A weak wreath can instead be used to equip st with a premonad structure, as defined by Böhm [14, Def. 2.1].

Definition 5.2.4 (Böhm)**.** A **premonad** on x is a triple (t, μ, η) consisting of a 1-cell $t\colon x \to x$ and 2-cells $\mu\colon t^2 \to t$, the multiplication, and $\eta\colon 1 \to t$, the preunit, such that the following diagrams commute.

$$\begin{tikzcd}
t^3 \arrow[r, "t\mu"] \arrow[d, "\mu t"'] \arrow[dr, phantom, "(t.a)"] & t^2 \arrow[d, "\mu"] \\
t^2 \arrow[r, "\mu"'] & t,
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[r, "t\eta"] \arrow[d, "\eta t"'] \arrow[dr, phantom, "(t.u)"] & t^2 \arrow[d, "\mu"] \\
t^2 \arrow[r, "\mu"'] & t,
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[rr, "\eta\eta"] \arrow[dr, "\eta"'] & (t.i) & t^2 \arrow[dl, "\mu"] \\
& t, &
\end{tikzcd}
\qquad
\begin{tikzcd}
t^2 \arrow[r, "\eta t^2"] \arrow[d, "\mu"'] \arrow[dr, phantom, "(t.e)"] & t^3 \arrow[d, "\mu t"] \\
t & t^2. \arrow[l, "\mu"]
\end{tikzcd}$$

A **premonad map** $(t, \mu^t, \eta^t) \to (s, \mu^s, \eta^s)$ is a 2-cell $\theta\colon t \to s$ compatible with the multiplication and the preunit, i.e. such that $(\theta.\mu)$ and $(\theta.\eta)$ commute.

As with monads, we will often use the 1-cell to refer to the entire premonad, so that (t, μ, η) will usually be abbreviated to t. It is easy to see that every monad is in particular a premonad.

Remark 5.2.5. Premonads on x and their maps form a category over $\mathcal{K}(x, x)$. In fact, examining the axioms involved in Definition 5.2.4, it is clear that this is the category of algebras in $\mathcal{K}(x, x)$ of a suitable 'premonoid' operad. As with all such categories, if the 2-cell of $\mathcal{K}$ underlying an idempotent map of premonads splits, then there is a unique lift of this splitting to the category of premonads. In other words, if $\theta\colon t \to t$ is an idempotent map of premonads on x which splits in $\mathcal{K}(x, x)$ as $\pi\colon t \to \widetilde{t}$ followed by $\iota\colon \widetilde{t} \to t$, then $\widetilde{t}$ admits a unique premonad structure such that π and ι are premonad maps given by

$$\widetilde{\mu} := \widetilde{t}^2 \xrightarrow{\iota\iota} t^2 \xrightarrow{\mu^t} t \xrightarrow{\pi} \widetilde{t} \qquad \text{and} \qquad \widetilde{\eta} := 1 \xrightarrow{\eta} t \xrightarrow{\pi} \widetilde{t}.$$

Theorem 5.2.6 (Böhm)**.** *Let t be a monad on x and s be an endo-1-cell of x. There is a one-to-one correspondence between*

(i) *weak wreaths of t over s, i.e. monads $(t \xrightarrow{(s,\delta)} t, \nu, \eta)$ in $\mathsf{EM}^w(\mathcal{K})$;*

(ii) *premonads (st, μ, η) in $\mathcal{K}$ such that the following commutes:*

$$\begin{tikzcd}
stst^2 \arrow[r, "\mu t"] \arrow[d, "sts\mu^t"'] & st^2 \arrow[d, "s\mu^t"] \\
stst \arrow[r, "\mu"'] & st.
\end{tikzcd}$$

The main interest of premonads and, in particular, weak wreaths is that, subject to the splitting of a certain idempotent 2-cell, they correspond exactly to the endo-1-cells which admit a retract with a monad structure, as the following lemma shows. The main result of part (i) is given by Böhm [14, Lem. 2.2], but we shall need the slightly sharper statement.

Lemma 5.2.7. *Let $t\colon x \to x$ be a 1-cell in $\mathcal{K}$.*

(i) *If* (t, μ, η) *is a premonad, then* $t \xrightarrow{\eta t} t^2 \xrightarrow{\mu} t$ *is an idempotent map of premonads. If the underlying 2-cell of this idempotent splits, say through* $\pi\colon t \to \widetilde{t}$ *and* $\iota\colon \widetilde{t} \to t$*, then* $\widetilde{t}$ *can be equipped with a unique premonad structure such that* π *and* ι *are premonad maps, given by*

$$\widetilde{\mu} := \widetilde{t}^2 \xrightarrow{\iota\iota} t^2 \xrightarrow{\mu} t \xrightarrow{\pi} \widetilde{t} \qquad \textit{and} \qquad \widetilde{\eta} := 1 \xrightarrow{\eta} t \xrightarrow{\pi} \widetilde{t}.$$

Moreover, $(\widetilde{t}, \widetilde{\mu}, \widetilde{\eta})$ *is actually a monad.*

(ii) *If* $\iota\colon \widetilde{t} \to t$ *is a retract in* $\mathcal{K}(x, x)$*, with retraction* $\pi\colon t \to \widetilde{t}$*, and* $(\widetilde{t}, \widetilde{\mu}, \widetilde{\eta})$ *is a monad, then* t *can be equipped with a unique premonad structure such that* ι *and* π *are premonad maps and* $\widetilde{t}$ *is the monad arising from* t *by part* (i)*, given by*

$$\mu := t^2 \xrightarrow{\pi\pi} \widetilde{t}^2 \xrightarrow{\widetilde{\mu}} \widetilde{t} \xrightarrow{\iota} t \qquad \textit{and} \qquad \eta := 1 \xrightarrow{\widetilde{\eta}} \widetilde{t} \xrightarrow{\iota} t.$$

(iii) *The two processes in* (i) *and* (ii) *are mutually inverse.*

Proof. For (i), first note that we have

$$\mu = (\mu \cdot \eta t) \cdot \mu \tag{5.3}$$

by

$$\begin{tikzcd}
t \arrow[r, "\eta t"] & t^2 \arrow[rr, equal] \arrow[drr, phantom, "(t.a)"] & & t^2 \arrow[d, "\mu"] \\
t^2 \arrow[u, "\mu"] \arrow[r, "\eta t^2"] \arrow[ur, phantom, "(*)"] \arrow[rrr, bend right, "\mu"'] & t^3 \arrow[u, "t\mu"'] \arrow[r, "\mu t"] \arrow[r, phantom, shift right=4, "(t.e)"] & t^2 \arrow[r, "\mu"'] & t.
\end{tikzcd}$$

It is clear then that the 2-cell $\mu \cdot \eta t$ is idempotent. It is compatible with the preunits by $(t.i)$, and the multiplication by

$$\begin{tikzcd}
t^2 \arrow[r, "\eta t^2"] \arrow[dd, "\mu"'] \arrow[ddr, phantom, "(*)"] & t^3 \arrow[rr, "t^2\eta t"] \arrow[dd, "t\mu"'] \arrow[dr, "t\eta t^2"] & & t^4 \arrow[r, "t^2\mu"] \arrow[d, "t\mu t"'] \arrow[dl, phantom, "(t.u)"] \arrow[dr, phantom, "(t.a)"] & t^3 \arrow[r, "\mu t"] \arrow[dd, "t\mu"'] \arrow[ddr, phantom, "(t.a)"] & t^2 \arrow[dd, "\mu"] \\
& & t^4 \arrow[r, "t\mu t"'] \arrow[d, phantom, "(t.e)"] & t^3 \arrow[dr, "t\mu"'] & & \\
t \arrow[r, "\eta t"'] & t^2 \arrow[rrr, equal] & \, & & t^2 \arrow[r, "\mu"'] & t.
\end{tikzcd}$$

By Remark 5.2.5, we split the idempotent $\mu \cdot \eta t$ to get a unique premonad structure $(\widetilde{t}, \widetilde{\mu}, \widetilde{\eta})$ such that $\pi\colon t \to \widetilde{t}$ and $\iota\colon \widetilde{t} \to t$ are premonad maps.

Since ι is a premonad map, we have $\iota \cdot \widetilde{\mu} \cdot \widetilde{\eta}\widetilde{t} = \mu \cdot \eta t \cdot \iota = \iota$, so $\widetilde{\mu} \cdot \widetilde{\eta}\widetilde{t} = 1$ because ι is monic. As $\mu \cdot t\eta = \mu \cdot \eta t$ by $(t.u)$, we similarly have $\widetilde{\mu} \cdot \widetilde{t}\widetilde{\eta} = 1$. Hence, $\widetilde{t}$ is actually a monad.

For (ii), first note that π and ι are compatible with the multiplication and preunit defined in the statement (since $\pi \cdot \iota = 1$), so they will become premonad maps as soon as we show that (t, μ, η) is a premonad. Moreover, $\mu = (\iota \cdot \pi) \cdot \mu$ and $\eta = (\iota \cdot \pi) \cdot \eta$ by definition. The axioms of a premonad for t then follow easily from those of $\tilde{t}$, e.g. $(t.u)$ follows from

$$\begin{aligned}
\mu \cdot t\eta &= \iota \cdot \pi \cdot \mu \cdot t\eta \\
&= \iota \cdot \tilde{\mu} \cdot \tilde{t}\tilde{\eta} \cdot \pi && \text{by } (\pi.\mu) \text{ and } (\pi.\eta) \\
&= \iota \cdot \tilde{\mu} \cdot \tilde{\eta}\tilde{t} \cdot \pi && \text{by } (\tilde{t}.u) \\
&= \iota \cdot \pi \cdot \mu \cdot \eta t && \text{by } (\pi.\mu) \text{ and } (\pi.\eta) \\
&= \mu \cdot \eta t.
\end{aligned}$$

Since $(\tilde{t}, \tilde{\mu}, \tilde{\eta})$ is a monad, the 2-cells in the second and third line are just $\iota \cdot \pi$, so that the idempotent $\mu \cdot \eta t$ is $\iota \cdot \pi$. The monad structure induced by (i) on $\tilde{t}$ is easily seen to agree with its original one, since $\pi \cdot \iota = \tilde{t}$.

Now suppose (t, μ', η') is a premonad structure on t such that π and ι are premonad maps and $\mu' \cdot \eta' t = \iota \cdot \pi$. Since ι is compatible with the preunits, we immediately have $\eta' = \iota \cdot \tilde{\eta} = \eta$. Moreover,

$$\mu' \overset{(5.3)}{=} (\iota \cdot \pi) \cdot \mu' \overset{(\pi.\mu)}{=} \iota \cdot \tilde{\mu} \cdot \pi\pi = \mu.$$

Hence, (t, μ, η) is the unique premonad structure on t with these properties.

For (iii), we have already shown that passing from a monad structure on a retract $\tilde{t}$ to a premonad structure on t and back results in the original monad structure. If (t, μ, η) is a premonad inducing the retract monad $(\tilde{t}, \tilde{\mu}, \tilde{\eta})$, we obtain a new multiplication and preunit for t as

$$\mu' := t^2 \xrightarrow{\iota\iota \cdot \pi\pi} t^2 \xrightarrow{\mu} t \xrightarrow{\iota \cdot \pi} t \qquad \text{and} \qquad \eta' := 1 \xrightarrow{\eta} t \xrightarrow{\iota \cdot \pi} t.$$

However, these coincide with the old multiplication and preunit since $\iota \cdot \pi = \mu \cdot \eta t$,

$$\begin{aligned}
\mu' &= \iota \cdot \pi \cdot \mu \cdot \iota\iota \cdot \pi\pi \\
&= \iota \cdot \pi \cdot \iota \cdot \pi \cdot \mu && \text{by } ((\iota \cdot \pi).\mu) \\
&= \mu && \text{by } (5.3),
\end{aligned}$$

and $\iota \cdot \pi \cdot \eta = \eta$ by $(t.i)$. ☐

Thanks to the previous lemma and Theorem 5.2.6, a weak wreath, and in particular a π-weak distributive law, induces a monad as soon as a certain idempotent 2-cell in $\mathcal{K}$ splits. We will call this monad the **weak composite** induced by the weak wreath. We record its construction as a corollary.

Corollary 5.2.8. *Let $\delta \colon ts \to st$ be a π-weak distributive law. Then the 2-cell*

$$st \xrightarrow{\eta^t st} tst \xrightarrow{\delta t} st^2 \xrightarrow{s\mu^t} st \tag{5.4}$$

is idempotent. Moreover, if $st \xrightarrow{\pi} \widetilde{st} \xrightarrow{\iota} st$ is a splitting of this idempotent, then *the 2-cells*

$$\widetilde{\eta} := 1 \xrightarrow{\eta^s\eta^t} st \xrightarrow{\pi} \widetilde{st}$$

$$\widetilde{\mu} := \widetilde{st}\widetilde{st} \xrightarrow{\iota\iota} stst \xrightarrow{s\delta t} s^2t^2 \xrightarrow{\mu^s\mu^t} st \xrightarrow{\pi} \widetilde{st}$$

make $(\widetilde{st}, \widetilde{\mu}, \widetilde{\eta})$ *a monad, which we denote by* $s \,\widetilde{\circ}_\delta\, t$.

Proof. A π-weak distributive law is in particular a weak wreath, so we can use Theorem 5.2.6 to construct a premonad structure on st. Examining the proof of [14, Thm. 2.3], we see that the multiplication and preunit are

$$\eta := 1 \xrightarrow{\eta^t\eta^s} ts \xrightarrow{\delta} st,$$

$$\mu := stst \xrightarrow{s\delta t} s^2t^2 \xrightarrow{\mu^s\mu^t} st \xrightarrow{\eta^t st} tst \xrightarrow{\delta s} st^2 \xrightarrow{s\mu^t} st.$$

The corresponding idempotent of Lemma 5.2.7(i) is then $\mu \cdot \eta st$ which simplifies to (5.4) by

$$\begin{tikzcd}
st \arrow[r, "\eta^t\eta^s st"] \arrow[dr, bend right, "\eta^t st"'] \arrow[dr, phantom, "(s.l)"] & ts^2t \arrow[r, "\delta st"] \arrow[d, "t\mu^s t"] & stst \arrow[r, "s\delta t"] \arrow[dr, phantom, "(\delta.\mathrm{w}\mu^s)"] & s^2t^2 \arrow[r, "\mu^s t^2"] & st^2 \arrow[r, "s\mu^t"] \arrow[d, "\eta^t st^2"'] & st \arrow[dd, "\eta^t st"] \\
 & tst \arrow[r, "\delta t"'] & st^2 \arrow[d, "s\mu^t"'] \arrow[dr, phantom, "(t.a)"] & st^3 \arrow[l, "s\mu^t t"'] \arrow[d, "st\mu^2"] & tst^2 \arrow[l, "\delta t^2"'] \arrow[dr, phantom, "(*)"] & \\
 & & st & st^2 \arrow[l, "s\mu^t"] & & tst. \arrow[ll, "\delta t"]
\end{tikzcd}$$

The retract monad is that given by the same lemma. The unit $\widetilde{\eta}$ in the statement already coincides with the one there since

$$\begin{aligned}
\iota \cdot \pi \cdot \delta \cdot \eta^t\eta^s &= \iota \cdot \pi \cdot \delta \cdot t\eta^s \cdot \eta^t \\
&= \iota \cdot \pi \cdot \iota \cdot \pi \cdot \eta^s t \cdot \eta^t && \text{by } (\delta.\mathrm{w}\eta^s) \\
&= \iota \cdot \pi \cdot \eta^s\eta^t
\end{aligned}$$

and ι is (split) monic. To see that the multiplication also agrees, simply note that $\mu = \iota \cdot \pi \cdot \mu^s\mu^t \cdot s\delta t$. ☐

So far, all our statements have been about monads in $\mathcal{K}$. We should expect corresponding statements about their Eilenberg–Moore objects, when they exist. This is achieved by the pseudofunctor $J^w \colon \mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$ constructed by Böhm [14, Thm 3.5] under the assumption that $\mathcal{K}$ admits the construction of algebras and splittings for all its idempotent 2-cells. It sends a monad t on x to its Eilenberg–Moore object x^t. Given a 1-cell $(g, \varphi) \colon (x, t) \to (y, s)$, the 2-cell

$$gu^t \xrightarrow{\eta^s gu^t} sgu^t \xrightarrow{\varphi u^t} gtu^t \xrightarrow{gu^t\epsilon^t} gu^t$$

is idempotent, and hence splits through some $\pi^g\colon gu^t \to \widetilde{g}$ and $\iota^g\colon \widetilde{g} \to gu^t$. Then

$$\widetilde{\varphi} := s\widetilde{g} \xrightarrow{s\iota} sgu^t \xrightarrow{\varphi u^t} gtu^t \xrightarrow{gu^t\epsilon^t} gu^t \xrightarrow{\iota} \widetilde{g}$$

is such that $(\widetilde{g}, \widetilde{\varphi})$ is a lax monad functor $(x^t, x^t) \to (y, s)$, and hence induces a 1-cell $\overline{g}\colon x^t \to y^s$ such that $u^s\overline{g} = \widetilde{g}$. This is how $J^w(g, \varphi)$ is defined. We will not need the explicit action of J^w on the 2-cells. In any case, being a pseudofunctor, J^w sends monads to monads, so that a weak wreath on t induces a monad on x^t.

Since our primary interest is on $\mathsf{Mnd}^\pi(\mathcal{K})$, we focus on this process in this restricted case.

Definition 5.2.9. For $\mathcal{K}$ admitting the construction of algebras, let $\mathsf{Lift}^\pi(\mathcal{K})$ be the 2-category where

- 0-cells are monads in $\mathcal{K}$,
- 1-cells $(x, t) \to (y, s)$ are tuples $(g, \overline{g}, \pi^g, \iota^g)$ fitting into a diagram

$$\begin{tikzcd}
x^t \arrow[r, "\overline{g}"] \arrow[d, "u^t"'] & y^s \arrow[d, "u^s"] \\
x \arrow[r, "g"'] \arrow[ur, phantom, "{\overset{\pi^g}{\Nearrow}\ \underset{\iota^g}{\Swarrow}}"] & y
\end{tikzcd}$$

  in $\mathcal{K}$, where $\pi^g \cdot \iota^g = 1$,

- 2-cells $(g, \overline{g}, \pi^g, \iota^g) \to (h, \overline{h}, \pi^h, \iota^h)$ are pairs $(\alpha, \overline{\alpha})$ where $\alpha\colon g \to h$ and $\overline{\alpha}\colon \overline{g} \to \overline{h}$ are 2-cells in $\mathcal{K}$ such that

$$\begin{tikzcd}
x^t \arrow[r, "\overline{g}"'] \arrow[r, bend left=70, "\overline{h}"] \arrow[r, phantom, bend left=35, "\Uparrow\overline{\alpha}"] \arrow[d, "u^t"'] & y^s \arrow[d, "u^s"] & & x^t \arrow[r, "\overline{h}"] \arrow[d, "u^t"'] & y^s \arrow[d, "u^s"] \\
x \arrow[r, "g"'] \arrow[ur, phantom, "\Nearrow_{\pi^g}"] & y & = & x \arrow[r, "h"] \arrow[r, bend right=70, "g"'] \arrow[r, phantom, bend right=35, "\Uparrow\alpha"] \arrow[ur, phantom, "\pi^h\Nearrow"] & y.
\end{tikzcd}$$

Composition of 1-cells is by horizontal pasting, and composition of 2-cells is done componentwise. Altogether, $\mathsf{Lift}^\pi(\mathcal{K})$ admits two forgetful 2-functors $U, \overline{U}\colon \mathsf{Lift}^\pi(\mathcal{K}) \to \mathcal{K}$, with

$$U(g, \overline{g}, \pi^g, \iota^g) = g \qquad \text{and} \qquad \overline{U}(g, \overline{g}, \pi^g, \iota^g) = \overline{g}.$$

Note that even though the section ι^g does not play a role in the definition of the 2-cells, it cannot be dropped without changing $\mathsf{Lift}^\pi(\mathcal{K})$, since it differentiates 1-cells with the same retraction but different sections.

Theorem 5.2.10 (Böhm)**.** *Let $\mathcal{K}$ admit the construction of algebras and splittings of idempotent 2-cells. The restriction of the pseudofunctor $J^w\colon \mathsf{EM}^w(\mathcal{K}) \to \mathcal{K}$ to $\mathsf{Mnd}^\pi(\mathcal{K})$ factors through $\overline{U}\colon \mathsf{Lift}^\pi(\mathcal{K}) \to \mathcal{K}$, and this factorisation is in fact a biequivalence*

$$\mathsf{Mnd}^\pi(\mathcal{K}) \simeq \mathsf{Lift}^\pi(\mathcal{K}).$$

Proof. This is essentially the content of Böhm [14, Thm. 4.4]. It only remains to check that the pseudofunctor $\mathsf{Mnd}^\pi(\mathcal{K}) \to \mathsf{Lift}^\pi(\mathcal{K})$ is bi-essentially surjective, in the terminology of Lack [39, p. 122]. This holds by construction, because both 2-categories have the same 0-cells. ☐

5.3 Weak composite monads

This section is devoted to characterising weak and π-weak distributive laws in terms of the weak composite monads they induce by Corollary 5.2.8. Its main results can be thought of as a generalisation of Theorem 2.3.13 to the weak setting. The purpose of this generalisation from the point of view of pushforwards of lifted monads should be clear from Section 5.1, where Theorem 2.3.13 was extensively used to relate them to distributive laws.

The first step is to show that the canonical 2-cells from s and t to the weak composite induced by a π-weak distributive law between them are monad maps.

Lemma 5.3.1. *Let s and t be monads on x, and $\delta\colon ts \to st$ be a π-weak distributive law. Let $\widetilde{st}$ be the weak composite monad δ induced by Corollary 5.2.8, with splitting $\pi\colon st \to \widetilde{st}$ and $\iota\colon \widetilde{st} \to st$. Then the 2-cells*

$$\sigma := s \xrightarrow{s\eta^t} st \xrightarrow{\pi} \widetilde{st} \tag{σ}$$

and

$$\tau := t \xrightarrow{\eta^s t} st \xrightarrow{\pi} \widetilde{st} \tag{τ}$$

are monad maps.

Proof. It is convenient to note that δ factors through $\iota\colon \widetilde{st} \to st$ and that composing with $\pi\colon st \to \widetilde{st}$ 'fixes' the commutativity of the diagrams that would make δ a genuine distributive law, i.e.

$$\iota \cdot \pi \cdot \delta = \delta, \tag{5.5}$$

$$\pi \cdot \delta \cdot \eta^t s = \pi \cdot s\eta^t, \tag{5.6}$$

$$\pi \cdot \delta \cdot t\eta^s = \pi \cdot \eta^s t, \tag{5.7}$$

$$\pi \cdot \delta \cdot t\mu^s = \pi \cdot \mu^s t \cdot s\delta \cdot \delta s. \tag{5.8}$$

The first one follows from

$$\begin{tikzcd}
ts \arrow[r, "\eta^t ts"] \arrow[rr, bend left=50, equal, "(t.l)"'] \arrow[d, "\delta"'] \arrow[dr, phantom, "(*)"] & t^2s \arrow[r, "\mu^t s"] \arrow[d, "t\delta"'] \arrow[drr, phantom, "(\delta.\mu^t)"] & ts \arrow[r, "\delta"] & st \\
st \arrow[r, "\eta^t st"'] & tst \arrow[rr, "\delta t"'] & & st^2. \arrow[u, "s\mu^t"']
\end{tikzcd}$$

Equation (5.6) then follows from

$$\begin{tikzcd}
s \arrow[r, "\eta^t s"] \arrow[d, "s\eta^t"'] \arrow[dr, phantom, "(*)"] & ts \arrow[r, "\delta"] & st \arrow[d, "st\eta^t"'] \arrow[drr, equal, bend left] \arrow[dr, phantom, "(t.r)"] & & \\
st \arrow[r, "\eta^t st"'] & tst \arrow[r, "\delta t"'] & st^2 \arrow[r, "s\mu^t"'] & st, &
\end{tikzcd}$$

since ι is monic and

$$\iota \cdot \pi \cdot s\eta^t = \delta \cdot \eta^t s = \iota \cdot \pi \cdot \delta \cdot \eta^t s.$$

Equations (5.7) and (5.8) are proved similarly, using (δ.wη^s) and (δ.wμ^s), respectively. We also record that, since π and $\iota \cdot \pi$ are premonad maps by Lemma 5.2.7,

$$\widetilde{\mu} \cdot \pi\pi = \pi \cdot \iota \cdot \pi \cdot \mu^s\mu^t \cdot s\delta t = \pi \cdot \mu^s\mu^t \cdot s\delta t, \tag{5.9}$$

and

$$\iota \cdot \pi \cdot \mu^s\mu^t \cdot s\delta t = \iota \cdot \pi \cdot \iota \cdot \pi \cdot \mu^s\mu^t \cdot s\delta t = \iota \cdot \pi \cdot \mu^s\mu^t \cdot s\delta t \cdot \iota\iota \cdot \pi\pi$$

so that, as ι is monic,

$$\pi \cdot \mu^s\mu^t \cdot s\delta t = \pi \cdot \mu^s\mu^t \cdot s\delta t \cdot \iota\iota \cdot \pi\pi. \tag{5.10}$$

That σ and τ are compatible with the unit is immediate from their definition and that of $\widetilde{\eta}$ in Corollary 5.2.8. We have that τ is compatible with the multiplication by

$$\begin{tikzcd}
stst \arrow[rr, "st(\iota\cdot\pi)"] & \arrow[d, phantom, "(\delta.\mathrm{w}\eta^s)"] & stst \arrow[rr, equal] & \arrow[d, phantom, "(*)"] & stst \arrow[r, "\pi\pi"] \arrow[d, "s\delta t"] & \widetilde{st}^2 \arrow[ddd, "\widetilde{\mu}"] \\
t^2 \arrow[u, "\eta^s t\eta^s t"] \arrow[r, "t^2\eta^s"] \arrow[d, "\mu^t"'] \arrow[dr, phantom, "(*)"] & t^2 s \arrow[r, "t\delta"] \arrow[d, "\mu^t s"] & tst \arrow[u, "\eta^s tst"'] \arrow[r, "\delta t"] \arrow[d, phantom, "(\delta.\mu^t)"] & st^2 \arrow[r, "\eta^s st^2"] \arrow[d, "s\mu^t"'] \arrow[dr, phantom, "(s.l)"] & s^2t^2 \arrow[d, "\mu^s\mu^t"] \arrow[r, phantom, "(5.9)"] & \\
t \arrow[r, "t\eta^s"] \arrow[d, "\eta^s t"'] & ts \arrow[rr, "\delta"] & \arrow[d, phantom, "(5.7)"] & st \arrow[r, equal] & st \arrow[dr, "\pi"'] & \\
st \arrow[rrrrr, "\pi"'] & & & & & \widetilde{st}.
\end{tikzcd}$$

Similarly, σ is compatible with the multiplication by

$$\begin{tikzcd}
stst \arrow[rr, "(\iota\cdot\pi)st"] \arrow[from=1-1, to=2-3, phantom, "\text{(5.6)}"] & & stst \arrow[rr, equal] \arrow[from=1-3, to=2-5, phantom, "(*)"] & & stst \arrow[r, "\pi\pi"] \arrow[d, "s\delta t"] & \widetilde{st}^2 \arrow[ddd, "\widetilde{\mu}"] \\
s^2 \arrow[u, "s\eta^t s\eta^t"] \arrow[r, "\eta^t s^2"] \arrow[d, "\mu^s"'] \arrow[from=2-1, to=3-2, phantom, "(*)"] & ts^2 \arrow[r, "\delta s"] \arrow[d, "t\mu^s"] \arrow[from=2-2, to=3-4, phantom, "\text{(5.8)}"] & sts \arrow[u, "sts\eta^t"] \arrow[r, "s\delta"] & s^2t \arrow[r, "s^2t\eta^t"] \arrow[d, "\mu^s t"'] \arrow[from=2-4, to=3-5, phantom, "(t.r)"] & s^2t^2 \arrow[d, "\mu^s\mu^t"] \arrow[r, phantom, "\text{(5.9)}"] & \\
s \arrow[r, "\eta^t s"] \arrow[d, "s\eta^t"'] \arrow[from=3-1, to=4-4, phantom, "\text{(5.6)}"] & ts \arrow[r, "\delta"] & st \arrow[dr, "\pi"'] & st \arrow[r, equal] \arrow[d, "\pi"'] \arrow[from=3-4, to=4-6, phantom, "(=)"] & st \arrow[dr, "\pi"] & \\
st \arrow[rrr, "\pi"'] & & & \widetilde{st} \arrow[rr, equal] & & \widetilde{st}.
\end{tikzcd}$$

$\square$

Even in the case of (nonweak) composite monads and distributive laws, a monad structure on st such that $\eta^s t\colon t \to st$ and $s\eta^t\colon s \to st$ are monad maps is not enough to guarantee that st is the monad induced by a distributive law. What is missing is Beck's middle unitary law (see Theorem 2.3.13). Condition (ii)(a) in the next theorem is the analogue of this law for π-weak distributive laws. In fact, the addition of only one more condition, namely (ii)(b), is enough to completely characterise π-weak distributive laws of t over s in terms of monad structures on retracts of st.

Theorem 5.3.2. *Let s and t be monads on x. There is a one-to-one correspondence between*

(i) *π-weak distributive laws $\delta\colon ts \to st$ such that (5.4) splits;*

(ii) *retracts $\widetilde{st} \underset{\pi}{\overset{\iota}{\rightleftarrows}} st$ with a monad structure $(\widetilde{st}, \widetilde{\mu}, \widetilde{\eta})$ such that*

$$\sigma := s \xrightarrow{s\eta^t} st \xrightarrow{\pi} \widetilde{st} \qquad \textit{and} \qquad \tau := t \xrightarrow{\eta^s t} st \xrightarrow{\pi} \widetilde{st}$$

are monad maps and the following commute:

$$\begin{tikzcd}
& st \arrow[dl, "\sigma\tau"'] \arrow[dr, "\pi"] \arrow[d, phantom, "\text{(a)}"] & \\
\widetilde{st}^2 \arrow[rr, "\widetilde{\mu}"'] & \, & \widetilde{st},
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{st}t \arrow[rr, "\iota t"] \arrow[d, "\widetilde{st}\tau"'] & \text{(b)} & st^2 \arrow[d, "s\mu^t"] \\
\widetilde{st}^2 \arrow[r, "\widetilde{\mu}"'] & \widetilde{st} \arrow[r, "\iota"'] & st.
\end{tikzcd}$$

Proof. First we show how the data of (i) give rise to those of (ii). The monad structure on a retract of st is that of Corollary 5.2.8. We have already seen that σ and τ are monad maps in Lemma 5.3.1.

To show that (a) and (b) commute, first note that

$$\begin{tikzcd}
st^2 \arrow[r, "\eta^t st^2"] \arrow[d, "s\mu^t"'] \arrow[rrr, bend left, "(\iota\cdot\pi)t"] \arrow[rrr, phantom, bend left=20, "(\iota\cdot\pi)"] & tst^2 \arrow[r, "\delta t^2"] \arrow[d, phantom, "(*)"] & st^3 \arrow[r, "s\mu^t t"] \arrow[d, "st\mu^t"'] \arrow[rd, phantom, "(t.a)"] & st^2 \arrow[d, "s\mu^t"] \\
st \arrow[r, "\eta^t st"'] \arrow[rrr, bend right, "\iota\cdot\pi"'] \arrow[rrr, phantom, bend right=20, "(\iota\cdot\pi)"] & tst \arrow[r, "\delta t"'] & st^2 \arrow[r, "s\mu^t"'] & st,
\end{tikzcd} \tag{5.11}$$

and

$$\begin{tikzcd}
 & s^2t \arrow[r, "s\eta^t st"] \arrow[dd, "\eta^t s^2t"'] \arrow[ldddd, bend right, "\mu^s t"'] \arrow[rrr, bend left, "s(\iota\cdot\pi)"] \arrow[rrr, phantom, bend left=15, "(\iota\cdot\pi)"] & stst \arrow[r, "s\delta t"] \arrow[d, "\iota\iota\cdot\pi\pi"] & s^2t^2 \arrow[r, "s^2\mu^t"] \arrow[d, phantom, "(5.10)"] & s^2t \arrow[r, "\mu^s t"] & st \arrow[d, "\pi"] \\
 & \arrow[r, phantom, "(5.6)"] & stst \arrow[r, "s\delta t"] & s^2t^2 \arrow[r, "\mu^s\mu^t"] \arrow[d, phantom, "(5.10)"] & st \arrow[r, "\pi"] & \widetilde{st} \arrow[d, equal] \\
\arrow[r, phantom, "(*)"] & ts^2t \arrow[r, "\delta st"] \arrow[dd, "t\mu^s t"] & stst \arrow[r, "s\delta t"] \arrow[u, "\iota\iota\cdot\pi\pi"] & s^2t^2 \arrow[r, "\mu^s\mu^t"] \arrow[d, "\mu^s t^2"'] \arrow[rd, phantom, "(*)"] & st \arrow[d, equal] \arrow[r, "\pi"] & \widetilde{st} \arrow[dd, equal] \\
 & \arrow[r, phantom, "(\delta.\mathrm{w}\mu^s)"] & & st^2 \arrow[r, "s\mu^t"] \arrow[d, "(\iota\cdot\pi)t"'] \arrow[rd, phantom, "(5.11)"] & st \arrow[d, "\iota\cdot\pi"] \arrow[r, phantom, "(\pi\cdot\iota)"] & {} \\
st \arrow[r, "\eta^t st"] \arrow[rrrrr, bend right, "\pi"'] \arrow[rrrrr, phantom, bend right=10, "(\iota\cdot\pi)"] & tst \arrow[rr, "\delta t"] & & st^2 \arrow[r, "s\mu^t"] & st \arrow[r, "\pi"] & \widetilde{st}
\end{tikzcd} \tag{5.12}$$

commute. Then (a) commutes by

$$\begin{tikzcd}
 & stst \arrow[r, "s\delta t"] \arrow[rrr, bend left, "\pi\pi"] \arrow[rd, phantom, "(\iota\cdot\pi)"] & s^2t^2 \arrow[d, "s^2\mu^t"] \arrow[rr, phantom, "(5.9)"] & & \widetilde{st}^2 \arrow[d, "\widetilde{\mu}"] \\
 & s^2t \arrow[u, "s\eta^t st"] \arrow[r, "s(\iota\cdot\pi)"] \arrow[d, "\mu^s t"] \arrow[ld, phantom, "(s.r)"] & s^2t \arrow[r, "\mu^s t"] \arrow[d, phantom, "(5.12)"] & st \arrow[r, "\pi"] & \widetilde{st} \arrow[d, equal] \\
st \arrow[r, equal] \arrow[ru, bend left, "s\eta^s t"] & st \arrow[rrr, "\pi"'] & & & \widetilde{st}.
\end{tikzcd}$$

Lastly, (b) commutes since the outside face of

$$\begin{tikzcd}
st^2 \arrow[rr, "\pi t"] \arrow[dr, "\sigma\tau\tau"] \arrow[dd, "s\mu^t"'] & & \widetilde{st}t \arrow[r, "\iota t"] \arrow[d, "\widetilde{st}\tau"] & st^2 \arrow[dd, "s\mu^t"] \\
& \widetilde{st}^3 \arrow[r, "\widetilde{\mu}\widetilde{st}"] \arrow[d, "\widetilde{st}\widetilde{\mu}"'] \arrow[l, phantom, "(\tau.\mu)"] \arrow[ur, phantom, "\text{(a)}"] \arrow[dr, phantom, "(\widetilde{st}.a)"] & \widetilde{st}^2 \arrow[d, "\widetilde{\mu}"] & \\
st \arrow[r, "\sigma\tau"] \arrow[rr, bend right, "\pi"', "\text{(a)}"] & \widetilde{st}^2 \arrow[r, "\widetilde{\mu}"] & \widetilde{st} \arrow[r, "\iota"] & st
\end{tikzcd}$$

commutes by (5.11), and πt is (split) epic.

Next, we show how the data of (ii) can be used to construct those of (i). We define

$$\delta := ts \xrightarrow{\tau\sigma} \widetilde{st}^2 \xrightarrow{\widetilde{\mu}} \widetilde{st} \xrightarrow{\iota} st,$$

and show that it satisfies the axioms of a π-weak distributive law (Definition 5.2.3). By definition,

$$\pi \cdot \delta = \widetilde{\mu} \cdot \tau\sigma. \tag{5.13}$$

Then $(\delta.\mu^t)$ follows from

$$\begin{tikzcd}
t^2s \arrow[rr, "t\delta"] \arrow[dr, "\tau\tau\sigma"] \arrow[dd, "\mu^t s"'] & & tst \arrow[r, "\tau\sigma t"] \arrow[d, "\tau\pi"'] \arrow[rrr, bend left, "\delta t", "(\delta)"'] & \widetilde{st}^2t \arrow[r, "\widetilde{\mu}t"] \arrow[d, "\widetilde{st}^2\tau"'] & \widetilde{st}t \arrow[r, "\iota t"] \arrow[d, "\widetilde{st}\tau"] & st^2 \arrow[dd, "s\mu^t"] \\
& \widetilde{st}^3 \arrow[r, "\widetilde{st}\widetilde{\mu}"] \arrow[d, "\widetilde{\mu}\widetilde{st}"'] \arrow[l, phantom, "(\tau.\mu)"] \arrow[ur, phantom, "(5.13)"] \arrow[dr, phantom, "(\widetilde{st}.a)"] & \widetilde{st}^2 \arrow[d, "\widetilde{\mu}"'] \arrow[ur, phantom, "\text{(a)}"] & \widetilde{st}^3 \arrow[l, "\widetilde{st}\widetilde{\mu}"'] \arrow[r, "\widetilde{\mu}\widetilde{st}"] \arrow[d, phantom, "(\widetilde{st}.a)"] \arrow[ur, phantom, "(*)"] & \widetilde{st}^2 \arrow[d, "\widetilde{\mu}"] \arrow[r, phantom, "\text{(b)}"] & \\
ts \arrow[r, "\tau\sigma"'] \arrow[rrrrr, bend right, "\delta"', "(\delta)"] & \widetilde{st}^2 \arrow[r, "\widetilde{\mu}"'] & \widetilde{st} \arrow[rr, equal] & & \widetilde{st} \arrow[r, "\iota"'] & st.
\end{tikzcd}$$

For the remaining two axioms, first note that

$$\begin{tikzcd}
st \arrow[r, equal] \arrow[d, "\widetilde{\eta}\sigma t"] \arrow[dd, bend right=60, "\eta^t st"', "(\tau.\eta)"] \arrow[dr, phantom, "(\widetilde{st}.l)"] & st \arrow[rr, equal] \arrow[d, "\sigma"] & & st \arrow[d, "\pi"] \\
\widetilde{st}^2t \arrow[r, "\widetilde{\mu}t"] & \widetilde{st}t \arrow[r, "\widetilde{st}\tau"] \arrow[d, "\iota t"'] & \widetilde{st}^2 \arrow[r, "\widetilde{\mu}"] \arrow[u, phantom, "\text{(a)}"] \arrow[d, phantom, "\text{(b)}"] & \widetilde{st} \arrow[d, "\iota"] \\
tst \arrow[u, "\tau\sigma t"] \arrow[r, "\delta t"'] \arrow[ur, phantom, "(\delta)"] & st^2 \arrow[rr, "s\mu^t"'] & & st.
\end{tikzcd} \tag{5.14}$$

Using this, we see that (δ.wη^s) commutes by

$$\begin{tikzcd}
 & t \arrow[r, equal] \arrow[d, "\tau\widetilde{\eta}"'] \arrow[dd, bend right=70, "t\eta^s"'] \arrow[dr, phantom, "(\widetilde{st}.r)"] & t \arrow[r, "\eta^s t"] \arrow[d, "\tau"] \arrow[dr, phantom, "(\tau)"] & st \arrow[dd, "\iota\cdot\pi"] \\
 {} \arrow[r, phantom, "(\sigma.\eta)"] & \widetilde{st}^2 \arrow[r, "\widetilde{\mu}"'] \arrow[dr, phantom, "(\delta)"] & \widetilde{st} \arrow[dr, "\iota"] & \\
 & ts \arrow[u, "\tau\sigma"'] \arrow[rr, "\delta"'] & & st,
\end{tikzcd}$$

and (δ.wμ^s) commutes by

$$\begin{tikzcd}
ts^2 \arrow[rr, "\delta s"] \arrow[dr, "\tau\sigma\sigma"] \arrow[ddd, "t\mu^s"'] \arrow[r, phantom, "(5.13)"] & & sts \arrow[rr, "s\delta"] \arrow[d, "\pi\sigma"'] \arrow[dr, "\sigma\tau\sigma"] \arrow[r, phantom, "(5.13)"] & & s^2t \arrow[rr, "\mu^s t"] \arrow[d, "\sigma\pi"'] \arrow[dr, "\sigma\sigma\tau"] \arrow[r, phantom, "(\sigma.\mu)"] & & st \arrow[d, "\sigma\tau"] \arrow[dd, bend left=60, "\pi"] \\
{} \arrow[r, phantom, "(\sigma.\mu)"] & \widetilde{st}^3 \arrow[r, "\widetilde{\mu}st"'] \arrow[d, "st\widetilde{\mu}"'] & \widetilde{st}^2 \arrow[d, "\widetilde{\mu}"'] \arrow[r, phantom, "(a)"] & \widetilde{st}^3 \arrow[l, "\widetilde{\mu}st"] \arrow[r, "st\widetilde{\mu}"'] & \widetilde{st}^2 \arrow[d, "\widetilde{\mu}"'] \arrow[r, phantom, "(a)"] & \widetilde{st}^3 \arrow[l, "st\widetilde{\mu}"] \arrow[r, "\widetilde{\mu}st"'] & \widetilde{st}^2 \arrow[d, "\widetilde{\mu}"] \arrow[d, phantom, shift left=6, "(a)"] \\
 & \widetilde{st}^2 \arrow[r, "\widetilde{\mu}"'] \arrow[ur, phantom, "(\widetilde{st}.a)"] & \widetilde{st} \arrow[rr, equal] \arrow[ur, phantom, "(\widetilde{st}.a)"] & & \widetilde{st} \arrow[rr, equal] \arrow[ur, phantom, "(\widetilde{st}.a)"] & & \widetilde{st} \arrow[d, "\iota"] \\
ts \arrow[ur, "\tau\sigma"'] \arrow[rrrrrr, "\delta"'] \arrow[rrrr, phantom, "(\delta)", shift left=4] & & & & & & st.
\end{tikzcd}$$

It remains to show that the two processes described are inverse to each other. Starting with a π-weak distributive law δ produces a new π-weak distributive law

$$\delta' := ts \xrightarrow{\eta^s ts\eta^t} stst \xrightarrow{\iota\iota\cdot\pi\pi} stst \xrightarrow{s\delta t} s^2t^2 \xrightarrow{\mu^s\mu^t} st \xrightarrow{\iota\cdot\pi} st.$$

But

$$\begin{aligned}
\delta' &= \iota\cdot\pi\cdot\mu^s\mu^t\cdot s\delta t\cdot\iota\iota\cdot\pi\pi\cdot\eta^s ts\eta^t \\
&= \iota\cdot\pi\cdot\mu^s\mu^t\cdot s\delta t\cdot\eta^s ts\eta^t && \text{by (5.10)} \\
&= \iota\cdot\pi\cdot\mu^s\mu^t\cdot\eta^s st\eta^t\cdot\delta && \text{by } (*) \\
&= \iota\cdot\pi\cdot\delta && \text{by (5.5)} \\
&= \delta.
\end{aligned}$$

In the other direction, the data of (ii) induces a π-weak distributive law $\delta = \iota\cdot\widetilde{\mu}\cdot\tau\sigma$. The corresponding idempotent in (5.4) is again $\iota\cdot\pi$ by (5.14). This induces a monad structure on $\widetilde{st}$ given by

$$\widetilde{\eta}' := \pi\cdot\delta\cdot\eta^t\eta^s = \widetilde{\mu}\cdot\tau\sigma\cdot\eta^t\eta^s = \widetilde{\mu}\cdot\widetilde{\eta}\widetilde{\eta} = \widetilde{\eta},$$

since τ and σ are compatible with the unit, and

$$\widetilde{\mu}' := \pi\cdot\mu^s\mu^t\cdot s\delta t\cdot\iota\iota.$$

Note that

$$\widetilde{\mu}\cdot\sigma\tau\cdot\iota=\pi\cdot\iota=\widetilde{st}. \tag{5.15}$$

Then

$$\begin{aligned}
\widetilde{\mu}' &= \pi\cdot\mu^s\mu^t\cdot s\delta t\cdot\iota\iota \\
&= \pi\cdot\mu^s\mu^t\cdot s\iota t\cdot s\widetilde{\mu}t\cdot s\tau\sigma t\cdot\iota\iota \\
&= \widetilde{\mu}\cdot\sigma\tau\cdot\mu^s\mu^t\cdot s\iota t\cdot s\widetilde{\mu}t\cdot s\tau\sigma t\cdot\iota\iota && \text{by (a)} \\
&= \widetilde{\mu}\cdot\widetilde{\mu}\widetilde{\mu}\cdot\sigma\sigma\tau\tau\cdot s\iota t\cdot s\widetilde{\mu}t\cdot s\tau\sigma t\cdot\iota\iota && \text{by } (\sigma.\mu) \text{ and } (\tau.\mu) \\
&= \widetilde{\mu}\cdot\widetilde{\mu}\widetilde{st}\cdot\widetilde{st}\widetilde{\mu}\widetilde{st}\cdot\sigma\sigma\tau\tau\cdot s\iota t\cdot s\widetilde{\mu}t\cdot s\tau\sigma t\cdot\iota\iota && \text{by } (\widetilde{st}.a) \\
&= \widetilde{\mu}\cdot\widetilde{\mu}\widetilde{st}\cdot\sigma\widetilde{st}\tau\cdot s\widetilde{\mu}t\cdot s\tau\sigma t\cdot\iota\iota && \text{by (5.15)} \\
&= \widetilde{\mu}\cdot\widetilde{\mu}\widetilde{st}\cdot\widetilde{st}\widetilde{\mu}\widetilde{st}\cdot\sigma\tau\sigma\tau\cdot\iota\iota && \text{by } (*) \\
&= \widetilde{\mu}\cdot\widetilde{\mu}\widetilde{\mu}\cdot\sigma\tau\sigma\tau\cdot\iota\iota && \text{by } (\widetilde{st}.a) \\
&= \widetilde{\mu} && \text{by (5.15)},
\end{aligned}$$

so we recover the original monad structure on $\widetilde{st}$. □

We would like a similar characterisation of weak distributive laws in terms of weak composites. The following rewriting lemma will be useful for this. It implies that if $\delta\colon ts\to st$ is a weak distributive law, then a composite of whiskerings of δ, μ^t and μ^s by s and t whose codomain is st is uniquely determined by its domain. Faces of diagrams which commute by Lemma 5.3.3 will be labelled by (coh).

Lemma 5.3.3. *Let $\mathcal{M}$ be the 2-category generated by a single object, two endo-1-cells s and t, and 2-cells*

- $\mu^s\colon s^2\to s$ *satisfying* $\mu^s\cdot s\mu^s=\mu^s\cdot\mu^s s$,
- $\mu^t\colon t^2\to t$ *satisfying* $\mu^t\cdot t\mu^t=\mu^t\cdot\mu^t t$,
- $\delta\colon ts\to st$ *such that* $(\delta.\mu^s)$ *and* $(\delta.\mu^t)$ *commute,*

and no other relations. Let w be a word in s and t with at least one copy of each. Then $\mathcal{M}$ has a unique 2-cell of type $w\to st$.

Proof. A 1-cell f of $\mathcal{M}$ is a (possibly empty) word in s and t. Let $l(f)$ denote the length of the word. Let $p(f)$ be the number of pairs (i,j) with $1\le i<j\le l(f)$ such that the i- and j-th letters of f are t and s respectively. Let $q(f)=l(f)+p(f)$. We induct on $q(w)$.

The 2-cells of $\mathcal{M}$ are composites of whiskerings of μ^s, μ^t and δ by s and t. We shall call such whiskerings **basic 2-cells**. If α is a basic 2-cell, then $q(\operatorname{dom}\alpha)>q(\operatorname{cod}\alpha)$. It follows that $q(\operatorname{dom}\alpha)>q(\operatorname{cod}\alpha)$ for any nonidentity 2-cell in $\mathcal{M}$.

The base case is $q(w)=2$. If $l(w)\le 1$ then $p(w)=0$, so we must have $l(w)=2$ and $p(w)=0$. This implies that $w=st$. By the above observation, the only 2-cell $st\to st$ is the identity.

Now let $q(w)\ge 3$. If $l(w)=2$, we must have $w=ts$, and again it is clear that the only 2-cell $ts\to st$ is δ. Otherwise, $l(w)\ge 3$. Let $w=x_1x_2\dots x_{l(w)}$

with $x_i \in \{s, t\}$ for each i. If $p(w) > 0$, then there is some i such that $x_i = t$ and $x_{i+1} = s$, so we have

$$w = x_1 \dots ts \dots x_{l(w)} \xrightarrow{x_1 \dots \delta \dots x_{l(w)}} x_1 \dots st \dots x_{l(w)} \xrightarrow{!} st,$$

where ! is the unique 2-cell given by induction, since $q(x_1 \dots st \dots x_{l(w)}) < q(w)$. Otherwise, there is some i such that $x_i = x_{i+1}$ and we can do something similar with a whiskering of μ^{x_i}. This shows the existence of the required cell.

Any nonidentity 2-cell $\alpha\colon w \to st$ factors through some basic 2-cell $\beta\colon w \to w'$. We show that α does not depend on the choice of β. Let $\beta_1\colon w \to w_1$ and $\beta_2\colon w \to w_2$ be two basic 2-cells. Being whiskerings of μ^s, μ^t or δ, β_1 and β_2 act on two adjacent letters of w, say

$$\beta_1 = x_1 \dots x_{i_1 - 1}\gamma_1 x_{i_1+2} \dots x_{l(w)} \quad \text{and} \quad \beta_2 = x_1 \dots x_{i_2-1}\gamma_2 x_{i_2+2} \dots x_{l(w)}$$

where $\gamma_1, \gamma_2 \in \{\mu^s, \mu^t, \delta\}$. If $i_1 = i_2$ then $\beta_1 = \beta_2$, because the domains of μ^s, μ^t and δ are different. If $|i_1 - i_2| \geq 2$, then β_1 and β_2 affect disjoint sets of letters. In particular, there is a whiskering $\beta_1'\colon w_1 \to w_3$ of γ_1 and a whiskering $\beta_2'\colon w_2 \to w_3$ of γ_2 with the same codomain. Then

$$\begin{tikzcd}
w \arrow[r, "\beta_1"] \arrow[d, "\beta_2"'] \arrow[dr, phantom, "(*)"] & w_1 \arrow[d, "\beta_2'"] \arrow[ddr, bend left, "!"] & \\
w_2 \arrow[r, "\beta_1'"'] \arrow[drr, bend right, "!"'] & w_3 \arrow[dr, "!"] & \\
 & & st,
\end{tikzcd}$$

commutes by strong induction, so β_1 and β_2 result in the same 2-cell α.

Otherwise, $i_2 = i_1 + 1$ without loss of generality. A simple case analysis shows that $(\gamma_1, \gamma_2) \in \{(\mu^s, \mu^s), (\mu^t, \mu^t), (\mu^t, \delta), (\delta, \mu^s)\}$. In each case, the relations the generating 2-cells satisfy allow us to draw a diagram similar to the one above, which again commutes by strong induction. ☐

Theorem 5.3.4. *Let s and t be monads on the object $x \in \mathcal{K}$. There is a one-to-one correspondence between*

(i) *weak distributive laws $\delta\colon ts \to st$ such that (5.4) splits;*

(ii) *retracts $\widetilde{st} \underset{\pi}{\overset{\iota}{\rightleftarrows}} st$ with a monad structure $(\widetilde{st}, \widetilde{\mu}, \widetilde{\eta})$ satisfying the conditions of Theorem 5.3.2(ii) and such that the following commute:*

$$\begin{tikzcd}
 & t \arrow[dl, "\tau"'] \arrow[dr, "\eta^s t"] & \\
\widetilde{st} \arrow[rr, "\iota"'] & \text{(c)} & st,
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{st}s \arrow[r, "\iota s"] \arrow[d, "\widetilde{st}\sigma"'] & sts \arrow[r, "s\tau\sigma"] & s\widetilde{st}^2 \arrow[r, "s\widetilde{\mu}"] & s\widetilde{st} \arrow[r, "s\iota"] & s^2t \arrow[d, "\mu^s t"] \\
\widetilde{st}^2 \arrow[rr, "\widetilde{\mu}"'] & & \widetilde{st} \arrow[rr, "\iota"'] \arrow[u, phantom, "\text{(d)}"] & & st.
\end{tikzcd}$$

Proof. First, we show that the data of (i) give rise to those of (ii). Any weak distributive law is in particular a π-weak distributive law. By Theorem 5.3.2, we need only check that the two new diagrams commute.

We have

$$\begin{tikzcd}
t \arrow[r, "\eta^t t"] \arrow[d, "\eta^s t"'] \arrow[rrr, bend left=40, equal] \arrow[dr, phantom, "(*)"] & t^2 \arrow[r, equal, "(t.l)"] \arrow[d, "t\eta^s t"'] \arrow[dr, phantom, "(\delta.\eta^s)"] & t^2 \arrow[r, "\mu^t"] \arrow[d, "\eta^s t^2"] \arrow[dr, phantom, "(*)"] & t \arrow[d, "\eta^s t"] \\
st \arrow[r, "\eta^t st"'] \arrow[rrr, bend right=40, "\iota \cdot \pi"'] & tst \arrow[r, "\delta t"'] & st^2 \arrow[r, "s\mu^t"'] \arrow[d, phantom, "(\iota \cdot \pi)"] & st, \\
 & & &
\end{tikzcd}$$

so that $\iota \cdot \tau = \iota \cdot \pi \cdot \eta^s t = \eta^s t$, giving (c).

For the other diagram, first note that

$$\begin{tikzcd}
stst \arrow[rr, "s\delta t"] \arrow[d, "\eta^t st \eta^t st"'] \arrow[dr, "\eta^t stst"] \arrow[ddd, bend right=60, "(\iota \cdot \pi)(\iota \cdot \pi)"'] & & s^2t^2 \arrow[r, "\mu^s\mu^t"] \arrow[d, phantom, "(*)"] & st \arrow[d, "\eta^t st"'] \arrow[ddd, bend left=60, "\iota \cdot \pi"] \\
tst^2st \arrow[r, "ts\mu^t st"'] \arrow[d, "\delta t \delta t"] \arrow[u, phantom, "(t.r)"] \arrow[d, phantom, "(\iota \cdot \pi)"'] & tstst \arrow[r, "ts\delta t"'] & ts^2t^2 \arrow[r, "t\mu^s\mu^t"'] \arrow[d, phantom, "(\mathrm{coh})"] & tst \arrow[d, "\delta t"'] \arrow[d, phantom, "(\iota \cdot \pi)"] \\
st^2st^2 \arrow[d, "s\mu^t s\mu^t"] & & & st^2 \arrow[d, "s\mu^t"'] \\
stst \arrow[rr, "s\delta t"] & & s^2t^2 \arrow[r, "\mu^s\mu^t"] & st,
\end{tikzcd}$$

and hence the outside face of

$$\begin{tikzcd}
stst \arrow[r, "s\delta t"] \arrow[d, "\pi\pi"'] & s^2t^2 \arrow[r, "\mu^s\mu^t"] \arrow[d, phantom, "(5.9)"] & st \arrow[d, "\pi"] \\
\widetilde{st}^2 \arrow[rr, "\widetilde{\mu}"'] \arrow[d, "\iota\iota"'] & & \widetilde{st} \arrow[d, "\iota"] \\
stst \arrow[r, "s\delta t"'] & s^2t^2 \arrow[r, "\mu^s\mu^t"'] & st
\end{tikzcd}$$

commutes. Since $\pi\pi$ is (split) epic, the bottom pentagon also commutes, giving

$$\iota \cdot \widetilde{\mu} = \mu^s\mu^t \cdot s\delta t \cdot \iota\iota. \tag{5.16}$$

Then one sees that (d) commutes by precomposing the following diagram with the 2-cell ιs:

$$\begin{tikzcd}
 & sts \arrow[r, "(\iota\cdot\pi)s"] & sts \arrow[rr, "s\delta"] & & s^2t \arrow[dd, "\mu^s t"] \\
sts \arrow[ur, bend left, equal] \arrow[ur, phantom, "(t.r)"'] \arrow[r, "st\eta^t s"'] \arrow[d, equal] & st^2s \arrow[u, "s\mu^t s"'] \arrow[ur, phantom, "(5.11)"] \arrow[r, "(\iota\cdot\pi)ts"'] \arrow[d, phantom, "(5.6)"] & st^2s \arrow[u, "s\mu^t s"'] \arrow[d, "st\delta"] & (\mathrm{coh}) & \\
sts \arrow[r, "sts\eta^t"] \arrow[dr, bend right, "\pi\sigma"'] \arrow[dr, phantom, "(\sigma)"] & stst \arrow[r, "\iota\iota\cdot\pi\pi"] \arrow[d, "\pi\pi"] \arrow[dr, phantom, "(*)"] & stst \arrow[r, "s\delta t"] & s^2t^2 \arrow[r, "\mu^s\mu^t"] \arrow[d, phantom, "(5.16)"] & st \\
 & \widetilde{st}^2 \arrow[r, equal] & \widetilde{st}^2 \arrow[u, "\iota\iota"'] \arrow[rr, "\widetilde{\mu}"'] & & \widetilde{st}. \arrow[u, "\iota"']
\end{tikzcd}$$

Next, we show how the data of (ii) give rise to that of (i). Again, by Theorem 5.3.2 we get a π-weak distributive law

$$\delta := \quad ts \xrightarrow{\tau\sigma} \widetilde{st}\widetilde{st} \xrightarrow{\widetilde{\mu}} \widetilde{st} \xrightarrow{\iota} st. \tag{δ}$$

The axiom $(\delta.\eta^s)$ follows from

$$\begin{tikzcd}
ts \arrow[r, "\tau\sigma"] \arrow[d, "\delta"'] \arrow[dr, phantom, "(5.13)"] & \widetilde{st}^2 \arrow[d, "\widetilde{\mu}"'] \arrow[dr, phantom, "(\widetilde{st}.r)"] & t \arrow[l, "\tau\widetilde{\eta}"'] \arrow[d, "\tau"'] \arrow[ll, bend right, "t\eta^s"'] \arrow[l, phantom, bend right, "(\sigma.\eta)"'] \arrow[dr, bend left, "\eta^s t"] \arrow[dr, phantom, "(\mathrm{c})"'] & \\
st \arrow[r, "\pi"'] & \widetilde{st} \arrow[r, equal] & \widetilde{st} \arrow[r, "\iota"'] & st
\end{tikzcd}$$

together with (5.5). Finally, the axiom $(\delta.\mu^s)$ follows from

$$\begin{tikzcd}
ts^2 \arrow[r, "\tau\sigma s"] \arrow[ddd, "t\mu^s"'] \arrow[rrr, bend left, "\delta s"] \arrow[rrr, phantom, bend left=15, "(\delta)"] & \widetilde{st}^2 s \arrow[r, "\widetilde{\mu}s"] \arrow[d, "\widetilde{st}^2\sigma"'] \arrow[dr, phantom, "(*)"] & \widetilde{st}s \arrow[r, "\iota s"] \arrow[d, "\widetilde{st}\sigma"] & sts \arrow[r, "s\delta"] & s^2t \arrow[dd, "s\pi"] \\
(\sigma.\mu) & \widetilde{st}^3 \arrow[r, "\widetilde{\mu}\widetilde{st}"'] \arrow[d, "\widetilde{st}\widetilde{\mu}"'] \arrow[dr, phantom, "(\widetilde{st}.a)"] & \widetilde{st}^2 \arrow[d, "\widetilde{\mu}"] & (\mathrm{d}) & \\
 & \widetilde{st}^2 \arrow[r, "\widetilde{\mu}"'] \arrow[dr, phantom, "(\delta)"] & \widetilde{st} \arrow[d, "\iota"] & & s\widetilde{st} \arrow[d, "s\iota"] \\
ts \arrow[ur, "\tau\sigma"'] \arrow[rr, "\delta"'] & & st & & s^2t. \arrow[ll, "\mu^s t"]
\end{tikzcd}$$

These two processes are inverse to each other by Theorem 5.3.2. $\square$

5.4 Pushforwards and weak distributive laws

In this last section we use our characterisations of π-weak distributive laws to relate them to pushforwards of lifted monads much like we did for distributive laws in Section 5.1. We use the same notation as there, i.e. $g\colon x \to y$ is a 1-cell with codensity monad $g_\sharp 1$ and counit ϵ^x and $(g, \theta)\colon (x, \overline{s}) \to (y, s)$ is a lax monad functor. We abbreviate $g_\sharp 1$ to t for convenience.

The next proposition is the analogue of Proposition 5.1.2 for π-weak distributive laws.

Proposition 5.4.1. *Let $\delta\colon ts \to st$ be a π-weak distributive law such that (5.4) splits through $\pi\colon st \to \widetilde{st}$ and $\iota\colon \widetilde{st} \to st$, giving $\widetilde{st}$ a monad structure. If (5.1) commutes, then the unique 2-cell $\xi\colon \widetilde{st} \to g_\sharp \overline{s}$ such that*

$$\epsilon^{\overline{s}} \cdot \xi g = \theta \cdot s\epsilon^x \cdot \iota g \tag{ξ}$$

is a monad map.

Moreover, if θ and ϵ^x are invertible, and $(\iota \cdot \pi)g$ is the identity on stg, then (5.1) commutes.

Proof. We will repeatedly use the fact that

$$\theta \cdot s\epsilon^x \cdot (\iota \cdot \pi)g = \theta \cdot s\epsilon^x, \tag{5.17}$$

as shown by

$$\begin{tikzcd}
stg \arrow[r, "\eta^t stg"] \arrow[d, "s\epsilon^x"'] \arrow[rrr, bend left=40, "(\iota \cdot \pi)g"] \arrow[rrr, phantom, bend left=20, "(\iota \cdot \pi)"] & tstg \arrow[r, "\delta tg"] \arrow[d, "ts\epsilon^x"'] \arrow[dr, phantom, "(*)"] & st^2g \arrow[r, "s\mu^t g"] \arrow[d, "st\epsilon^x"'] \arrow[dr, phantom, "(\epsilon^x.\mu)"] & stg \arrow[d, "s\epsilon^x"] \\
sg \arrow[d, "\theta"'] \arrow[r, phantom, "(*)"] & tsg \arrow[r, "\delta g"'] \arrow[d, "tg\overline{s}"'] \arrow[drr, phantom, "(5.1)"] & stg \arrow[r, "s\epsilon^x"'] & sg \arrow[d, "\theta"] \\
g\overline{s} \arrow[r, "\eta^t g\overline{s}"] \arrow[rrr, equal, bend right=40, "(\epsilon^x.\eta)"] & tg\overline{s} \arrow[rr, "\epsilon^x \overline{s}"] & & g\overline{s}.
\end{tikzcd}$$

By Theorem 4.1.5, it suffices to show that $(g, \theta \cdot s\epsilon^x \cdot \iota g)$ is a lax transformation of monads. Indeed, it is compatible with the unit because

$$\begin{tikzcd}
g \arrow[r, "\eta^s \eta^t g"] \arrow[dr, bend right, "\widetilde{\eta}"'] \arrow[dr, phantom, "(\widetilde{\eta})"] & stg \arrow[r, "s\epsilon^x"] \arrow[d, "\pi g"] \arrow[drr, phantom, "(5.17)"] & sg \arrow[r, "\theta"] & g\overline{s} \\
& \widetilde{st}g \arrow[r, "\iota g"'] & stg \arrow[r, "s\epsilon^x"'] & sg \arrow[u, "\theta"']
\end{tikzcd}$$

commutes. It is compatible with the multiplication by

$$\begin{tikzcd}
\widetilde{st}\widetilde{st}g \arrow[r, "\widetilde{st}\iota g"] \arrow[d, "\iota\iota g"'] \arrow[dddd, bend right=60, "\widetilde{\mu}g"', "(\widetilde{\mu})"] & \widetilde{st}stg \arrow[r, "\widetilde{st}s\epsilon^x"] & \widetilde{st}sg \arrow[r, "\widetilde{st}\theta"] \arrow[d, phantom, "(*)"] & \widetilde{st}g\overline{s} \arrow[d, "\iota g\overline{s}"] \\
ststg \arrow[r, "sts\epsilon^x"] \arrow[d, "s\delta tg"'] \arrow[dr, phantom, "(*)"] & stsg \arrow[rr, "st\theta"] \arrow[d, "s\delta g"] \arrow[drr, phantom, "(5.1)"] & & stg\overline{s} \arrow[d, "s\epsilon^x\overline{s}"] \\
s^2t^2g \arrow[r, "s^2t\epsilon^x"] \arrow[d, "\mu^s\mu^t g"'] \arrow[dr, phantom, "(\epsilon^x.\mu)"] & s^2tg \arrow[r, "s^2\epsilon^x"] & s^2g \arrow[r, "s\theta"] \arrow[dl, "\mu^s g"'] \arrow[dr, phantom, "(\theta.\mu)"] & sg\overline{s} \arrow[d, "\theta\overline{s}"] \\
stg \arrow[r, "s\epsilon^x"'] \arrow[d, "\pi g"'] \arrow[dr, phantom, "(5.17)"] & sg \arrow[r, "\theta"'] & g\overline{s} & g\overline{s}^2 \arrow[l, "g\mu^{\overline{s}}"] \\
\widetilde{st}g \arrow[r, "\iota g"'] & stg \arrow[r, "s\epsilon^x"'] & sg. \arrow[u, "\theta"']
\end{tikzcd}$$

For the last statement, we use the same argument as for the last part of Proposition 5.1.2. The face labelled $(\delta.\eta^{g_\sharp n})$ there commutes in this case because

$$\begin{aligned}
\delta g \cdot \eta^t sg &= (\iota \cdot \pi \cdot \delta \cdot \eta^t s)g && \text{by (5.5)} \\
&= (\iota \cdot \pi \cdot s\eta^t)g && \text{by (5.6)} \\
&= s\eta^t g,
\end{aligned}$$

since we assumed that $(\iota \cdot \pi)g = stg$. ☐

Proposition 5.4.2. *Let $\iota\colon \widetilde{st} \to st$ be a retract, with retraction $\pi\colon st \to \widetilde{st}$, such that*

$$\begin{tikzcd}
stg \arrow[rr, "(\iota\cdot\pi)g"] \arrow[d, "s\epsilon^x"'] & & stg \arrow[d, "s\epsilon^x"] \\
sg \arrow[r, "\theta"'] & g\overline{s} & sg \arrow[l, "\theta"]
\end{tikzcd} \tag{5.18}$$

commutes. Let $\xi\colon \widetilde{st} \to g_\sharp\overline{s}$ be as in the previous proposition. If ξ is monic, the monad structure of $g_\sharp\overline{s}$ restricts along it and Theorem 5.3.2(ii)(b) commutes, then the restricted monad structure on $\widetilde{st}$ is the weak composite given by a π-weak distributive law of $g_\sharp 1$ over s satisfying (5.1).

Proof. We check the conditions of Theorem 5.3.2(ii). We already have that (b) commutes by assumption.

Let ν, φ and ψ be as in Section 5.1 and τ and ψ be as in Section 5.3. We show that $\xi \cdot \pi = \nu$, $\xi \cdot \tau = \varphi$ and $\xi \cdot \sigma = \psi$. Since ξ is monic, this makes τ and σ monad maps by Lemma 5.1.3.

Since $\epsilon^{\overline{s}}$ is the counit of a right extension, the first equation follows from

$$\epsilon^{\overline{s}} \cdot (\xi \cdot \pi)g \overset{(\xi)}{=} \theta \cdot s\epsilon^x \cdot \iota g \cdot \pi g \overset{(5.18)}{=} \theta \cdot s\epsilon^x \overset{(\nu)}{=} \epsilon^{\overline{s}} \cdot \nu g.$$

Then

$$\xi \cdot \tau \overset{(\tau)}{=} \xi \cdot \pi \cdot \eta^s t = \nu \cdot \eta^s t \qquad \text{and} \qquad \xi \cdot \sigma \overset{(\sigma)}{=} \xi \cdot \pi \cdot s\eta^t = \nu \cdot s\eta^t,$$

and we have

$$\epsilon^{\overline{s}} \cdot (\nu \cdot \eta^s t)g \overset{(\nu)}{=} \theta \cdot s\epsilon^x \cdot \eta^s tg \overset{(*)}{=} \theta \cdot \eta^s g \cdot \epsilon^x \overset{(\theta.\eta)}{=} g\eta^{\overline{s}} \cdot \epsilon^x \overset{(\varphi)}{=} \epsilon^{\overline{s}} \cdot \varphi g,$$
$$\epsilon^{\overline{s}} \cdot (\nu \cdot s\eta^t)g \overset{(\nu)}{=} \theta \cdot s\epsilon^x \cdot s\eta^t g \overset{(\epsilon^x.\eta)}{=} \theta \overset{(\psi)}{=} \epsilon^{\overline{s}} \cdot \psi g,$$

giving the other two.

The outside face of

$$\begin{tikzcd}
st \arrow[r, "\sigma\tau"] \arrow[dr, "\pi"'] & \widetilde{st}^2 \arrow[r, "\xi\xi"] \arrow[d, "\widetilde{\mu}"] \arrow[dr, phantom, "(\xi.\mu)"] & (g_\sharp\overline{s})^2 \arrow[d, "\mu^{g_\sharp\overline{s}}"] \\
 & \widetilde{st} \arrow[r, "\xi"'] & g_\sharp\overline{s}
\end{tikzcd}$$

commutes by our equations and Lemma 5.1.1. Since ξ is monic, this implies that the left triangle, i.e. Theorem 5.3.2(ii)(a), commutes.

Lastly, (5.1) commutes by

$$\begin{tikzcd}
tsg \arrow[r, "\tau\sigma g"] \arrow[d, equal] & \widetilde{st}^2 g \arrow[r, "\widetilde{\mu}g"] \arrow[d, "\xi\xi g"] \arrow[dr, phantom, "(\xi.\mu)"] & \widetilde{st}g \arrow[r, "\iota g"] \arrow[d, "\xi g"] & stg \arrow[ddd, "s\epsilon^x"] \\
tsg \arrow[r, "\varphi\psi g"] \arrow[d, "t\theta"'] \arrow[dr, phantom, "(\psi)"] & (g_\sharp\overline{s})^2 g \arrow[r, "\mu^{g_\sharp\overline{s}}g"] \arrow[d, "(g_\sharp\overline{s})\epsilon^{\overline{s}}"] & (g_\sharp\overline{s})g \arrow[dd, "\epsilon^{\overline{s}}"] \arrow[r, phantom, "(\xi)"] & {} \\
tg\overline{s} \arrow[r, "\varphi g\overline{s}"] \arrow[d, "\epsilon^x\overline{s}"'] \arrow[dr, phantom, "(\varphi)"] & (g_\sharp\overline{s})g\overline{s} \arrow[d, "\epsilon^{\overline{s}}\overline{s}"] \arrow[r, phantom, "(\epsilon^{\overline{s}}.\mu)"] & {} & \\
g\overline{s} \arrow[r, "g\eta^{\overline{s}}\overline{s}"'] \arrow[rr, bend right, equal, "(\overline{s}.l)"'] & g\overline{s}^2 \arrow[r, "g\mu^{\overline{s}}"'] & g\overline{s} & sg \arrow[l, "\theta"]
\end{tikzcd}$$

where the top composite is how δg is defined in Theorem 5.3.2, and the unlabelled face commutes by our earlier equations. □

To give an analogue of Theorem 5.1.5 for π-weak distributive laws we need the notion of a weak right extension. The word 'weak' here is taken in the sense of 'weak terminal object', which is an object of a category that admits a (not necessarily unique) morphism from any other object.

Definition 5.4.3. Let $g\colon x \to y$ and $f\colon x \to z$ be 1-cells in $\mathcal{K}$. A **weak right extension** of f along g is a weak terminal object of $g^\circ{\downarrow}f$. Explicitly, it is pair (h, ϵ) where $h\colon y \to z$ is a 1-cell and $\epsilon\colon hg \to f$ is a 2-cell such that for any other such pair (k, α) there exists some 2-cell $\widehat{\alpha}\colon k \to h$ such that $\epsilon \cdot \widehat{\alpha}g = \alpha$.

A 1-cell $l\colon z \to w$ **weakly preserves** (h, ϵ) if $(lh, l\epsilon)$ is in turn a weak extension of lf along g.

Of course, every right extension is a weak right extension. Just as weak terminal objects are not unique (e.g. any nonempty set is a weak terminal object of Set), neither are weak extensions. If a category $\mathcal{C}$ has a terminal object 1, then $x \in \mathcal{C}$ is a weak terminal object iff the unique morphism $x \to 1$ is a split epimorphism. Indeed, any morphism $1 \to x$ is a section.

Theorem 5.4.4. *Let $g\colon x \to y$ be a 1-cell in $\mathcal{K}$, whose codensity monad t exists and has counit $\epsilon^x\colon tg \to g$, and s be a monad on y whose 1-cell part weakly preserves $\operatorname{ran}_g g$. The data of*

(i) *a strong monad functor $(g, \theta)\colon (x, \bar{s}) \to (y, s)$ such that $g_\sharp \bar{s}$ exists and has counit $\epsilon^{\bar{s}}$;*

(ii) *a 2-cell $\iota\colon g_\sharp \bar{s} \to st$ which is a section in $g^\circ{\downarrow}g\bar{s}$ of the (necessarily split epic) unique morphism $\pi\colon (st, \theta \cdot s\epsilon^x) \to (g_\sharp \bar{s}, \epsilon^{\bar{s}})$, such that*

$$\begin{tikzcd}
(g_\sharp\bar{s})t \arrow[rr, "\iota t"] \arrow[d, "(g_\sharp\bar{s})(g_\sharp\eta^{\bar{s}})"'] & & st^2 \arrow[d, "s\mu^t"] \\
(g_\sharp\bar{s})^2 \arrow[r, "\mu^{g_\sharp\bar{s}}"'] & g_\sharp\bar{s} \arrow[r, "\iota"'] & st
\end{tikzcd} \tag{5.19}$$

commutes;

induce a π-weak distributive law δ of t over s satisfying (5.1) *such that the resulting weak composite monad is $g_\sharp \bar{s}$.*

Moreover, if ϵ^x is invertible and $(\iota \cdot \pi)g$ is the identity, the induced π-weak distributive law is the unique one such that the idempotent in (5.4) *is $\iota \cdot \pi$.*

Proof. The fact that $\iota \cdot \pi$ is a morphism in $g^\circ{\downarrow}g\bar{s}$ precisely means that (5.18) commutes. Moreover, the monad map ξ of Proposition 5.4.2 is the identity, since

$$\epsilon^{\bar{s}} \cdot \xi g \cdot \pi g \overset{(\xi)}{=} \theta \cdot s\epsilon^x \cdot \iota g \cdot \pi g = \theta \cdot s\epsilon^x = \epsilon^{\bar{s}} \cdot \pi g$$

and π is (split) epic. The last equation is the fact that π is a morphism $(st, \theta \cdot s\epsilon^x) \to (g_\sharp \bar{s}, \epsilon^{\bar{s}})$ in $g^\circ{\downarrow}g\bar{s}$. Proposition 5.4.2 then ensures that the monad structure on $g_\sharp \bar{s}$ is induced by a π-weak distributive law of t over s satisfying (5.1).

Now suppose ϵ^x is invertible and $(\iota \cdot \pi)g$ is the identity. Let $\delta\colon ts \to st$ be a π-weak distributive law such that the idempotent in (5.4) is $\iota \cdot \pi$. Since splitting of idempotents is unique up to (unique) isomorphism, we may take $\widetilde{st}$ in Proposition 5.4.1 to be $g_\sharp \bar{s}$ and $\xi\colon \widetilde{st} \to g_\sharp \bar{s}$ to be the identity. The same proposition ensures that this is a monad isomorphism, so that $g_\sharp \bar{s}$ is the weak composite induced by δ. By Theorem 5.3.2, the 2-cells ι and π and $g_\sharp \bar{s}$ with its pushforward monad structure uniquely determine δ. ☐

Example 5.4.5. We will use the previous theorem in Theorem 6.1.18 to construct a π-weak distributive law of the ultrafilter monad over the powerset monad, whose weak composite is the filter monad. In fact, this is the weak distributive law constructed by Garner in [28, §5.1]. In our setting, the fact that this is weak and not just π-weak is explained because the relevant section $\iota\colon \mathsf{F} \to \mathsf{PU}$ actually makes Theorem 5.3.4(ii)(c) and (d) commute.

Chapter 6

Examples of pushforward monads

In this final chapter we put to use the general theory developed in Chapters 4 and 5 to compute several examples of pushforward monads in the 2-category of categories. In fact, the patterns exhibited by these examples were the motivation for developing much of our results so far.

In Section 6.1, we tackle our most extensive source of examples: pushing monads forward along the inclusion $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$ of the category of finite sets into that of all sets. The most fundamental of these pushforwards, i.e. the codensity monad of J, is the ultrafilter monad U, whose category of algebras is the category of compact Hausdorff spaces and continuous maps between them. This already exemplifies how, even though the starting data is elementary, the resulting monads on Set can be quite interesting.

Monads on FinSet are rarer than one might expect at first. They correspond precisely to those finitary monads on Set which preserve finiteness. In the language of universal algebra, these are the locally finite varieties. We compute the pushforwards and the corresponding categories of algebras of several natural such monads (see Table 6.1). In particular, the pushforwards of the exception, reader, state, free M-set and powerset monads arise from distributive laws (weak in the case of the powerset monad) involving the ultrafilter monad. This provides examples for the results in Chapter 5. We also show that (with two trivial exceptions) the pushforward of a monad along J does not have rank (Theorem 6.1.24).

Section 6.2 categorifies some of the earlier examples by considering pushforwards along the inclusion $\mathsf{FinFam}(\mathcal{E}) \hookrightarrow \mathsf{Fam}(\mathcal{E})$ of the free finite-coproduct completion of a category $\mathcal{E}$ into its free coproduct completion. Leinster showed that the codensity monad of this inclusion, when $\mathcal{E}$ has coproducts and cofiltered limits, is the ultracoproduct monad[1]. We show that the pushforward of a particular lift of the powerset monad along the fibration $\mathsf{FinFam}(\mathcal{E}) \to \mathsf{FinSet}$ gives the reduced coproduct monad, dual to the construction of reduced products of algebras introduced by Frayne, Morel and Scott in [27].

Section 6.3 revisits the computation of the category of algebras of the codensity monad of the inclusion $\mathsf{Field} \hookrightarrow \mathsf{CRing}$. We already saw in Example 4.3.23(i) that this is the free product completion of Field using Diers's theory of multimonadic functors. We give a detailed elementary proof of this fact, which was

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[1] Leinster uses a dual convention, in which this is called the ultraproduct monad. What he calls $\mathsf{Fam}(\mathcal{E})$ is our $\mathsf{Fam}(\mathcal{E}^{\mathrm{op}})$.

T	FinSet^T	$J_\sharp T$	$\mathsf{Set}^{J_\sharp T}$	Reference
1	FinSet	U	CHaus	Theorem 6.1.5
T	1	T	1	Example 6.1.1
T_0	Ω	T_0	Ω	Example 6.1.2
D_2	$\mathsf{FinBool}$	D_2	$\mathsf{Set}^{\mathrm{op}}$	Example 6.1.3
V_k	fdVect_k	$\mathsf{Vect}_k(k^-, k)$	$\mathsf{Vect}_k^{\mathrm{op}}$	Example 6.1.4
E_E	E/FinSet	$\mathsf{E}_E\mathsf{U}$	E/CHaus	Example 6.1.8
M_M	$[\mathrm{B}M, \mathsf{FinSet}]$	$\mathsf{M}_M\mathsf{U}$	$[\mathrm{B}M, \mathsf{CHaus}]$	Example 6.1.9
R_I	$\{\varnothing\}((\mathsf{Set}_{\geq 1})^I)$	$\mathsf{R}_I\mathsf{U}$	$\{\varnothing\}((\mathsf{CHaus}_{\geq 1})^I)$	Example 6.1.10
S_S	FinSet	$\mathsf{S}_S\mathsf{U}$	CHaus	Example 6.1.11
P	$\mathsf{BddJoinSLat}$	F	$\mathsf{ContLat}$	Theorem 6.1.14

Table 6.1: Examples of pushforwards along $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$ of several monads T on FinSet and their categories of algebras (up to equivalence) computed in Section 6.1.

found independently.

Finally, Section 6.4 briefly considers the situation of pushing a monad forward along another monad. We compute the codensity monads and corresponding categories of algebras of the exception monads $- + E$ on Set.

6.1 Pushforwards from FinSet to Set

In this section, we explicitly compute the pushforward along the inclusion $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$ of various monads on FinSet. We will also show that, with the exception of two examples, all pushforwards along J have no rank (Theorem 6.1.24).

Note that J is a fully faithful, representably small functor into a complete category, so we are in the setting of Theorem 4.3.11. In particular, the problem of finding monads on FinSet reduces to that of finding monads on Set that restrict along J, i.e. that preserve finiteness. Moreover, since J is the inclusion of the finitely presentable objects of Set, we are in the situation of Remark 4.3.16. Hence, we can further reduce this problem to that of finding finitary monads on Set which preserve finiteness or, equivalently, finitary algebraic theories all of whose finitely generated free models are finite. In the language of universal algebra, these theories are known as the **locally finite varieties** and they have been studied, among others, by McKenzie and Valeriote [52]. A characterisation of these theories and their corresponding pushforwards would be a substantial undertaking, well beyond the scope of this thesis. Instead, we limit ourselves to studying several natural families of monads on FinSet and their pushforwards.

Since J is fully faithful, any monad (or even endofunctor) T on Set which preserves finiteness has, up to isomorphism, exactly one lift along J. We will denote it by $\overline{T}$. Because every monad on FinSet is the lift of some such monad on

Set, its name will almost always be written with an overline (omitted in Table 6.1 for readability).

There are at least four families of monads whose pushforward along J is easy to identify.

Example 6.1.1. Let T be the terminal monad on Set (Example 2.4.9). Then $\overline{\mathsf{T}}$ is the terminal monad on FinSet. It follows from the limit preservation property of Lemma 4.1.4 that $J_\sharp\overline{\mathsf{T}} \cong \mathsf{T}$. The categories of algebras of T and $\overline{\mathsf{T}}$ coincide: they are both equivalent to the terminal category.

Example 6.1.2. Let T_0 be the unique proper submonad of T (Example 2.4.10). For any nonempty set X, the limit formula in (4.6) for $J_\sharp\overline{\mathsf{T}}(X)$ and that for $J_\sharp\overline{\mathsf{T}_0}(X)$ agree, so we have $J_\sharp\overline{\mathsf{T}_0}(X) = 1$. Since J is fully faithful and $\varnothing \in \mathsf{FinSet}$, we have $J_\sharp\overline{\mathsf{T}_0}(\varnothing) = \varnothing$. Hence, $J_\sharp\overline{\mathsf{T}_0} \cong \mathsf{T}_0$. Once again, the categories of algebras of T_0 and $\overline{\mathsf{T}_0}$ coincide: they are both equivalent to the walking arrow category Ω.

Example 6.1.3. Let n be a finite set, and D_n be the endomorphism monad of n on Set (Example 2.4.15). Then $\overline{\mathsf{D}_n}$ is the codensity monad of $n\colon 1 \to \mathsf{FinSet}$ and D_n is that of Jn. Note that the right Kan extension $n_\sharp 1$ is pointwise, and its computation only involves finite limits. Since J preserves finite limits, it preserves $\mathrm{Ran}_n\, n$, and by Lemma 4.1.7 we have $J_\sharp\overline{\mathsf{D}_n} = J_\sharp(n_\sharp 1) \cong (Jn)_\sharp 1 = \mathsf{D}_n$.

The same argument shows that, for any functor $G\colon \mathcal{A} \to \mathsf{FinSet}$ from a finite category, and any monad T on $\mathcal{A}$, we have $J_\sharp(G_\sharp T) = (JG)_\sharp T$. Note that $G_\sharp T$ exists pointwise because FinSet has all finite limits.

Example 6.1.4. Let k be a finite field, and V_k be the free k-vector space monad on Set, which preserves finiteness. We have a commutative diagram of functors

$$\begin{tikzcd}
\mathsf{fdVect}_k \arrow[r, hook, "J_k"] \arrow[d, "U^{\overline{\mathsf{V}_k}}"'] & \mathsf{Vect}_k \arrow[d, "U^{\mathsf{V}_k}"] \\
\mathsf{FinSet} \arrow[r, hook, "J"'] & \mathsf{Set},
\end{tikzcd}$$

where the vertical arrows are monadic. Hence,

$$J_\sharp\overline{\mathsf{V}_k} \cong (JU^{\overline{\mathsf{V}_k}})_\sharp 1 \cong (U^{\mathsf{V}_k}J_k)_\sharp 1 \cong (U^{\mathsf{V}_k})_\sharp(J_{k\sharp}1)$$

by Lemma 4.1.7 and Remark 4.1.8. The monad $J_{k\sharp}1$ was shown by Leinster [45, Thm. 7.5] to be isomorphic to the double dualisation monad $(-)^{**}$ on Vect_k. Thus, the monad $J_\sharp\overline{\mathsf{V}_k}$ sends a set X to

$$U^{\mathsf{V}_k}(F^{\mathsf{V}_k}X)^{**} = \mathsf{Vect}_k([kX, k], k) = \mathsf{Vect}_k(k^X, k),$$

where k^X is the X-th power of k in Vect_k.

Leinster's Theorem 7.8 shows that the category of $(-)^{**}$-algebras is equivalent to $\mathsf{Vect}_k^{\mathrm{op}}$, which itself is equivalent to the category of linearly compact k-vector spaces. In fact, the proof shows that $(-)^*\colon \mathsf{Vect}_k^{\mathrm{op}} \to \mathsf{Vect}_k$ preserves all coequalisers, and hence satisfies the hypotheses of the Crude Monadicity Theorem. The latter is also true of U^{V_k} because V_k is finitary. Hence,

$$\mathsf{Vect}_k^{\mathrm{op}} \xrightarrow{(-)^*} \mathsf{Vect}_k \xrightarrow{U^{\mathsf{V}_k}} \mathsf{Set}$$

is monadic, and so the category of $J_\sharp\overline{\mathsf{V}_k}$-algebras is again equivalent to $\mathsf{Vect}_k^{\mathrm{op}}$. A direct proof of this fact using Borceux's [15, Vol. II, Thm. 4.4.5] suggested by Richard Garner involves noting that Vect_k is abelian and hence coexact, and that k is an injective regular cogenerator.

For the remainder of our examples, computing the relevant pointwise right Kan extension using (4.6) will be a significant part of the work needed to find a pushforward. Given a monad $\overline{T}$ on FinSet, we will write $\langle g\rangle \colon J_\sharp\overline{T}(X) \to J\overline{T}Y$ for the leg of the limit cone at $g\colon X \to JY \in X{\downarrow}J$. Which monad on FinSet this refers to will be clear from the context. We will often drop J from our notation, since it is simply a witness that Y is finite. The cone condition amounts to the fact that, if $f\colon Y \to Y'$ is a function between finite sets, then

$$\langle fg\rangle = \overline{T}f \cdot \langle g\rangle, \tag{6.1}$$

where we have already started to drop J. As demonstrated by this equation, we write fg or $f \cdot g$ for the composite of two functions f and g, reserving the latter notation for those cases where ambiguity might arise.

Since the ultrafilter monad will play a crucial rule in the computation of the remaining pushforwards, we provide a proof of its identification with $J_\sharp 1$. The reader may recall the necessary background on ultrafilters from Definition 2.4.18. Given a subset $A \subseteq X$, we write $[A]\colon X \to 2$ for the characteristic function of A, where $2 = \{0, 1\}$.

Theorem 6.1.5 (Kennison and Gildenhuys)**.** *The codensity monad of* $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$ *is the ultrafilter monad* U*.*

Proof. Since every ultrafilter on a finite set is principal, $\overline{\mathsf{U}} = J^\sharp\mathsf{U}$ is isomorphic to the identity. In particular, we have a map of monads $\varphi\colon \mathsf{U} \to J_\sharp 1$, which is the component at U of the unit of the adjunction in Theorem 4.3.11. Given an ultrafilter $\mathcal{U}$ on a set X and a function $g\colon X \to Y$ into a finite set, $\langle g\rangle\varphi_X(\mathcal{U})$ is the unique $y \in Y$ at which $\mathsf{U}g(\mathcal{U})$ is principal, i.e. the unique y such that $g^{-1}\{y\} \in \mathcal{U}$. We show that φ is invertible, and hence an isomorphism of monads.

Given $t \in J_\sharp 1(X)$, note that the following are equivalent for $A \subseteq X$:

(i) $\langle g\rangle(t) = y$ for all $g\colon X \to Y \in X{\downarrow}J$ and $y \in Y$ such that $g^{-1}\{y\} = A$;

(ii) $\langle g\rangle(t) = y$ for some $g\colon X \to Y \in X{\downarrow}J$ and $y \in Y$ such that $g^{-1}\{y\} = A$;

(iii) $\langle [A]\rangle(t) = 1$.

This is because if $g\colon X \to Y \in X{\downarrow}J$ and $y \in Y$ is such that $g^{-1}\{y\} = A$, then $[A] = [\{y\}]g$. By (6.1), we have $\langle [A]\rangle = [\{y\}]\langle g\rangle$ and so

$$\langle [A]\rangle(t) = 1 \iff \langle g\rangle(t) = y.$$

The equivalence between the three conditions then follows from the fact that the left-hand side does not depend on g or y.

The component of the inverse of φ at X sends $t \in J_\sharp 1(X)$ to the collection $\mathcal{U}_t$ of $A \subseteq X$ such that these equivalent conditions hold. We check that $\mathcal{U}_t$ is an ultrafilter.

First, taking g in (ii) to be $!\colon X \to 1$ shows that $X \in \mathcal{U}_t$. Given $A, B \subseteq X$, consider the function $([A], [B])\colon X \to 2^2$ and note that the preimage of $(1,1)$ is exactly $A \cap B$. Then

$$\begin{aligned} A \cap B \in \mathcal{U}_t &\iff \langle([A],[B])\rangle(t) = (1,1) && \text{by (i) and (ii)} \\ &\iff \langle[A]\rangle(t) = 1 \text{ and } \langle[B]\rangle(t) = 1 && \text{by (6.1)} \\ &\iff A \in \mathcal{U}_t \text{ and } B \in \mathcal{U}_t && \text{by (iii).} \end{aligned}$$

Lastly, note that $[X \setminus A] = \neg[A]$, where $\neg\colon 2 \to 2$ is logical negation. Hence, either $\langle[A]\rangle(t) = 1$ or $\langle[X \setminus A]\rangle(t) = 1$ but not both, giving the last ultrafilter axiom.

It remains to show that this indeed provides an inverse to φ_X. If $\mathcal{U}$ is an ultrafilter on X and $A \subseteq X$, then

$$A \in \mathcal{U}_{\varphi_X(\mathcal{U})} \iff \langle[A]\rangle(\varphi_X(\mathcal{U})) = 1 \iff [A]^{-1}\{1\} = A \in \mathcal{U}.$$

Conversely, if $t \in J_\sharp 1(X)$, $g\colon X \to Y \in X{\downarrow}J$ and $y \in Y$, then

$$\langle g\rangle(\varphi_X(\mathcal{U}_t)) = y \iff g^{-1}\{y\} \in \mathcal{U}_t \iff \langle g\rangle(t) = y. \qquad \square$$

Implicit in the previous proof is the idea, extensively used by Leinster in [45], that the value at X of $J_\sharp 1$ is the set of natural integration operators for finite-valued functions on X. This means that $t \in J_\sharp 1(X)$ gives a way of assigning, to each $g\colon X \to Y$ with Y finite, an integral $\int g\, dt \in Y$ with respect to t, which is $\langle g\rangle(t)$ in our notation. Condition (iii) in the proof then shows that such an integration operator is determined by a 2-valued measure on X, where the measure of $A \subseteq X$ is $\int [A]\, dt = \langle[A]\rangle(t) \in 2$. The measures obtained in this way are exactly the finitely additive ones (see Leinster's Lemma 3.1).

Recall from Example 2.4.21 that the category of U-algebras is CHaus equipped with the underlying set functor. Using the notation of Section 5.1, if T is a monad on Set which preserves finiteness, we have monad maps $\varphi\colon \mathsf{U} \to J_\sharp\overline{T}$ and $\psi\colon T \to J_\sharp\overline{T}$. In terms of categories of algebras, this translates to a commutative square of algebraic functors

$$\begin{tikzcd} \mathsf{Set}^{J_\sharp\overline{T}} \arrow[r, "U^\psi"] \arrow[d, "U^\varphi"'] & \mathsf{Set}^T \arrow[d, "U^T"] \\ \mathsf{CHaus} \arrow[r, "U^{\mathsf{U}}"'] & \mathsf{Set}. \end{tikzcd} \tag{6.2}$$

By Theorem 2.4.5, U^φ and U^ψ are monadic. Hence, regardless of whether there is some kind of distributive law of U over T, there is a monad on CHaus that T induces whose category of algebras is $\mathsf{Set}^{J_\sharp\overline{T}}$.

Of course, this phenomenon is quite specific to Set. In general, the presence of such a distributive law helps to understand the category of algebras of $J_\sharp\overline{T}$. We now give examples of monads for which such distributive laws are available.

6.1.1 Distributive laws under the ultrafilter monad

First we tackle those monads T on Set which preserve $\operatorname{Ran}_J J$ as well as finiteness. Theorem 5.1.5 guarantees the existence of a unique distributive law of U over T, whose induced composite monad is $J_\sharp\overline{T}$.

Corollary 6.1.6. *Let T be a monad on Set which preserves finiteness and cofiltered limits. There exists a unique distributive law of U over T. Moreover, the composite monad it induces is $J_\sharp\overline{T}$.*

Proof. We check the hypotheses of Theorem 5.1.5. Since FinSet has finite limits, the category $X{\downarrow}J$ is cofiltered for any set S, and hence the limit in (4.6) used to compute $J_\sharp 1(X)$ is cofiltered. By assumption, T preserves this limit, and hence $\operatorname{Ran}_J J$. Since T also preserves finiteness, we have a lift $\overline{T}$ of T to FinSet. Finally, the counit of $\operatorname{Ran}_J J$ is invertible by Lemma 2.2.7. ☐

The following standard result gives many examples of monads to which we can apply this corollary.

Lemma 6.1.7. *Any polynomial endofunctor of Set, i.e. one that can be written as*

$$\sum_{a\in A}(-)^{B_a}$$

for some sets A and $(B_a)_{a\in A}$, preserves connected limits, and hence cofiltered limits.

Proof. It suffices to show that $(-)^B\colon \mathsf{Set}\to\mathsf{Set}$ and $\sum\colon \mathsf{Set}^A\to\mathsf{Set}$ preserve connected limits for any sets A and B. The first is obvious, because $(-)^B$ is a right adjoint. The second is the well-known fact that, in Set, coproducts commute with connected limits. ☐

Example 6.1.8. Let E be a finite set and consider the exception monad $\mathsf{E}_E = - + E$ (Example 2.4.12). It preserves cofiltered limits by the previous lemma, since $- + E \cong - + \sum_{e\in E}(-)^{\varnothing}$. Hence, by Corollary 6.1.6, there is a unique distributive law $\delta\colon \mathsf{U}\mathsf{E}_E \to \mathsf{E}_E\mathsf{U}$ and $J_\sharp\overline{\mathsf{E}_E} \cong \mathsf{E}_E \circ_\delta \mathsf{U}$.

Given $\mathcal{U}\in\mathsf{U}(X+E)$, either $X\in\mathcal{U}$ or $E\in\mathcal{U}$ but not both. In the first case, $\{A\cap X \mid A\in\mathcal{U}\}$ is an ultrafilter on X, which is the value of $\delta_X(\mathcal{U})$. Otherwise, $\{A\cap E \mid A\in\mathcal{U}\}$ is an ultrafilter on E, which must be principal at some $e\in E$, and then $\delta_X(\mathcal{U}) = e$.

In fact, Manes and Mulry [51, Ex. 2.4.6] showed that there is a canonical distributive law of E_E over any monad T given by the map

$$\lambda_X\colon TX + E \to T(X+E)$$

which is $T\iota_1$ in the first component and $T\iota_2 \cdot \eta^T_E$ in the second. This gives an inverse to δ, so that $\mathsf{E}_E\mathsf{U} \cong \mathsf{U}\mathsf{E}_E$. This is part of the well-known fact that U preserves finite coproducts.

The corresponding unique lift of E_E to CHaus is unsurprising. The finite set E admits a unique compact Hausdorff topology, i.e. the discrete topology. Since CHaus has coproducts, the lift is simply given by the corresponding exception monad $-+E$ on CHaus. Its category of algebras, which necessarily coincides with that of $J_\sharp\overline{\mathsf{E}_E}$, is that of E-pointed compact Hausdorff spaces, which is equivalent to E/CHaus.

Example 6.1.9. Let M be a finite monoid and consider the free M-set monad $\mathsf{M}_M = M \times -$ (Example 2.4.16). Lemma 6.1.7 implies that M_M preserves cofiltered limits. Hence, there is a unique distributive law $\delta\colon \mathsf{U}\mathsf{M}_M \to \mathsf{M}_M\mathsf{U}$ and $J_\sharp\overline{\mathsf{M}_M} \cong \mathsf{M}_M \circ_\delta \mathsf{U}$.

Given $\mathcal{U} \in \mathsf{U}(M \times X)$, we must have $\{m\} \times X \in \mathcal{U}$ for exactly one $m \in M$. Then $\delta_X(\mathcal{U}) = (m, \pi_2\{A \cap (\{m\} \times X) \mid A \in \mathcal{U}\})$. As before, since U preserves finite coproducts, δ is invertible, and so δ^{-1} is also a distributive law.

The corresponding unique lift of M_M to CHaus is again unsurprising. The finite monoid M becomes a monoid in CHaus with the discrete topology, and the corresponding monad is that of actions of that monoid. Its category of algebras is isomorphic to the functor category $[\mathrm{B}M, \mathsf{CHaus}]$.

Example 6.1.10. Let I be a finite set and consider the reader monad $\mathsf{R}_I = (-)^I$ (Example 2.4.13). Once again, this monad preserves cofiltered limits by Lemma 6.1.7. Hence, there is a unique distributive law $\delta\colon \mathsf{U}\mathsf{R}_I \to \mathsf{R}_I\mathsf{U}$ and $J_\sharp\overline{\mathsf{R}_I} \cong \mathsf{R}_I \circ_\delta \mathsf{U}$.

Given $\mathcal{U} \in \mathsf{U}(X^I)$ and $i \in I$, we obtain an ultrafilter $\mathsf{U}\pi_i(\mathcal{U})$ on X. This is the i-th component of $\delta_X(\mathcal{U})$. Unlike in the previous two examples, δ is not invertible unless $I \in \{\varnothing, 1\}$. For $I = 2$, this follows from the fact that U is not commutative.

Note that R_I is the I-th power in $\mathsf{Mnd}_{\mathsf{Set}}$ of the identity monad, and similarly for $\overline{\mathsf{R}_I}$ in $\mathsf{Mnd}_{\mathsf{FinSet}}$. By Lemma 4.1.4, since J and hence $J_\circ\colon [\mathsf{FinSet}, \mathsf{FinSet}] \to [\mathsf{FinSet}, \mathsf{Set}]$ preserve finite limits, $J_\sharp\overline{\mathsf{R}_I}$ is the I-th power of U.

The finite set I with the discrete topology is an exponentiable object of CHaus simply because

$$\mathsf{CHaus}(X \times I, Y) \cong \mathsf{CHaus}(X, Y)^I \cong \mathsf{CHaus}(X, Y^I).$$

The lift of R_I to CHaus is then the corresponding reader monad on CHaus, i.e. the I-th power of the identity monad on CHaus. The same argument that Faro and Kelly give in [26, Prop. 11] applies in this case (even though their argument is given over Set, it works just as well over CHaus because the result of Barr they cite also holds there). Hence, the category of $J_\sharp\overline{\mathsf{R}_I}$ algebras is the result of freely adjoining an initial object to $(\mathsf{CHaus}_{\geq 1})^I$, the I-th power of the category of nonempty compact Hausdorff spaces.

Example 6.1.11. Let S be a finite set and consider the state monad $\mathsf{S}_S = (S \times -)^S$ (Example 2.4.14). It preserves cofiltered limits by Lemma 6.1.7. Thus, there is a unique distributive law $\delta\colon \mathsf{U}\mathsf{S}_S \to \mathsf{S}_S\mathsf{U}$ and $J_\sharp\overline{\mathsf{S}_S} \cong \mathsf{S}_S \circ_\delta \mathsf{U}$. The formula for this distributive law is built from that in the previous two examples.

The lift of S_S to CHaus is the corresponding state monad on CHaus, which exists because S is exponentiable. The composite of the right adjoints $(-)^S\colon \mathsf{CHaus} \to \mathsf{CHaus}$ and $U^{\mathsf{U}}\colon \mathsf{CHaus} \to \mathsf{Set}$ is exactly the S-th power of U^{U}, and it induces the monad $J_\sharp\overline{\mathsf{S}_S}$. Proposition 2.4.6 implies that $(U^{\mathsf{U}})^S$ is monadic, so that the category of algebras of $J_\sharp\overline{\mathsf{S}_S}$ is equivalent to CHaus.

We now turn our attention to the powerset monad P (Example 2.4.17), which clearly preserves finiteness. The functor P does not preserve $\mathrm{Ran}_J J$, but it does so weakly in the sense of Definition 5.4.3. First, we show that $J_\sharp\overline{\mathsf{P}}$ is isomorphic to the filter monad F of Example 2.4.20. The proof mimics that of Theorem 6.1.5.

Lemma 6.1.12. *The restriction of the filter monad* F *along* J *is* $\overline{\mathsf{P}}$.

Proof. For a set X, let $\sigma_X\colon \mathsf{P}X \to \mathsf{F}X$ be the map sending $A \subseteq X$ to the principal filter $\uparrow A$. It is natural in X because $\mathsf{F}f(\uparrow A) = \uparrow f(A)$ for any function $f\colon X \to Y$. In fact, it is a monad map. That it is compatible with the unit is immediate from the definition. For the multiplication, if $\mathfrak{A} \in \mathsf{P}^2 X$ then

$$(\sigma\sigma)_X(\mathfrak{A}) = \uparrow\{\uparrow A \mid A \in \mathfrak{A}\}$$

and

$$\begin{aligned} B \in (\mu^{\mathsf{F}} \cdot \sigma\sigma)_X(\mathfrak{A}) &\iff B^\sharp \in (\sigma\sigma)_X(\mathfrak{A}) \\ &\iff B^\sharp \supseteq \{\uparrow A \mid A \in \mathfrak{A}\} \\ &\iff \forall A \in \mathfrak{A},\ B \in \uparrow A \\ &\iff \forall A \in \mathfrak{A},\ B \supseteq A \\ &\iff B \supseteq \bigcup \mathfrak{A} = \mu^{\mathsf{P}}(\mathfrak{A}). \end{aligned}$$

Recall that if $\mathcal{F}$ is a filter on a finite set Y, then $\bigcap \mathcal{F} = \bigcap_{A \in \mathcal{F}} A$ is a finite intersection of elements of $\mathcal{F}$. This gives a least element of $\mathcal{F}$, at which $\mathcal{F}$ is principal. This shows that σJ is an isomorphism, so that F and P have the same restriction along J. □

This lemma ensures that we have a monad map $\psi\colon \mathsf{F} \to J_\sharp\overline{\mathsf{P}}$, the component at F of the unit of the adjunction in Theorem 4.3.11. We will use the next result to show that ψ is invertible.

Lemma 6.1.13. *Let* X *be a set and* $t \in J_\sharp\overline{\mathsf{P}}(X)$. *The following are equivalent for* $A \subseteq X$:

(i) $\langle g\rangle(t) \subseteq B$ *for all* $g\colon X \to Y \in X{\downarrow}J$ *and* $B \subseteq Y$ *such that* $g^{-1}B = A$;

(ii) $\langle g\rangle(t) \subseteq B$ *for some* $g\colon X \to Y \in X{\downarrow}J$ *and* $B \subseteq Y$ *such that* $g^{-1}B = A$;

(iii) $\langle [A]\rangle(t) \subseteq \{1\}$.

Proof. If $g \colon X \to Y \in X{\downarrow}J$ and $B \subseteq Y$ is such that $g^{-1}B = A$, then $[A] = [B]g$. By (6.1), we have $\langle [A] \rangle = \mathsf{P}[B] \cdot \langle g \rangle$ and so

$$\langle [A] \rangle(t) \subseteq \{1\} \iff \langle g \rangle(t) \subseteq B.$$

The equivalence between the three conditions then follows from the fact that the left-hand side does not depend on g or B. □

Theorem 6.1.14. *The pushforward $J_\sharp\overline{\mathsf{P}}$ is isomorphic to the filter monad* F.

Proof. Given a set X, a filter $\mathcal{F}$ on X, and $g \colon X \to Y \in X{\downarrow}J$, we get a filter $\mathsf{F}f(\mathcal{F})$ on Y, which is principal at $\bigcap \mathsf{F}f(\mathcal{F})$. The monad map $\psi \colon \mathsf{F} \to J_\sharp\overline{\mathsf{P}}$ is then determined by

$$\langle g \rangle \psi_X(\mathcal{F}) = \bigcap \mathsf{F}f(\mathcal{F}). \tag{6.3}$$

We now construct an inverse to ψ_X.

Given $t \in J_\sharp\overline{\mathsf{P}}$, we get a filter $\mathcal{F}_t$ on X by declaring that $A \in \mathcal{F}_t$ iff it satisfies any of the equivalent conditions of Lemma 6.1.13. Certainly, $X \in \mathcal{F}_t$ by taking g in condition (ii) to be $! \colon X \to 1$. Given $A, B \subseteq X$, consider the function $([A],[B]) \colon X \to 2^2$ and note that the preimage of $(1,1)$ is exactly $A \cap B$. Then

$$\begin{aligned} A \cap B \in \mathcal{F}_t &\iff \langle ([A],[B]) \rangle(t) \subseteq \{(1,1)\} && \text{by (i) and (ii)} \\ &\iff \langle [A] \rangle(t) \subseteq \{1\} \text{ and } \langle [B] \rangle(t) \subseteq \{1\} && \text{by (6.1)} \\ &\iff A \in \mathcal{F}_t \text{ and } B \in \mathcal{F}_t && \text{by (iii).} \end{aligned}$$

Hence, $\mathcal{F}_t$ is a filter on X.

Lastly, we check that this provides an inverse to ψ_X. Given $t \in J_\sharp\overline{\mathsf{P}}$, for $g \colon X \to Y \in X{\downarrow}J$, recall that $\langle g \rangle \psi_X(\mathcal{F}_t)$ is the smallest $B \subseteq Y$ such that $g^{-1}B \in \mathcal{F}_t$. But we have

$$g^{-1}B \in \mathcal{F}_t \iff \langle g \rangle(t) \subseteq B,$$

the forward implication coming from Lemma 6.1.13 condition (i), and the backwards one from condition (ii). Therefore, $\langle g \rangle \psi_X(\mathcal{F}_t) = \langle g \rangle(t)$, as needed. In the other direction, for $\mathcal{F} \in \mathsf{F}X$ and $A \subseteq X$ we have

$$\begin{aligned} A \in \mathcal{F}_{\psi_X(\mathcal{F})} &\iff \langle [A] \rangle \psi_X(\mathcal{F}) \subseteq \{1\} && \text{by (iii)} \\ &\iff \bigcap \mathsf{F}[A](\mathcal{F}) \subseteq \{1\} && \text{by (6.3)} \\ &\iff \{1\} \in \mathsf{F}[A](\mathcal{F}) \\ &\iff [A]^{-1}\{1\} = A \in \mathcal{F} && \text{by (2.11).} \end{aligned}$$

□

Remark 6.1.15. It follows from the isomorphism in the previous theorem that a collection $\mathcal{F}$ of subsets of X is a filter iff it induces a natural nondeterministic integration operator for finite valued functions on X. Given a function $g \colon X \to Y$ into a finite set, its nondeterministic integral with respect to $\mathcal{F}$ is $\int g \, d\mathcal{F} = \langle g \rangle \psi_X(\mathcal{F}) \subseteq Y$.

Using Theorem 5.4.4, we can show that $\mathsf{F} \cong J_\sharp\overline{\mathsf{P}}$ is in fact the weak composite induced by a π-weak distributive law of U over P. This π-weak distributive law is not new; it is the weak distributive law constructed by Garner [28, §5.1] using the theory of extensions of endofunctors and natural transformations from Set to Rel, which is the Kleisli category of P. In our context, it arises not mainly due to properties of P and its Kleisli category, but due to U being the codensity monad of J.

Proposition 6.1.16. *The endofunctor* P *weakly preserves* $\operatorname{Ran}_J J$*.*

Proof. As discussed after Definition 5.4.3, it suffices to show that the comparison natural transformation $\pi\colon \mathsf{P} \circ \operatorname{Ran}_J J \to \operatorname{Ran}_J \mathsf{P}J$ has a section ι such that $\mathsf{P}\epsilon^J \cdot \iota J = \epsilon^{\mathsf{P}J}$, where ϵ^J and $\epsilon^{\mathsf{P}J}$ are the counits of $\operatorname{Ran}_J J$ and $\operatorname{Ran}_J \mathsf{P}J$, respectively. Translating this along the isomorphisms of Theorems 6.1.5 and 6.1.14, π becomes the natural transformation $\mathsf{PU} \to \mathsf{F}$ which sends a set of ultrafilters to their intersection. The counits $\epsilon^J\colon \mathsf{U}J \to J$ and $\epsilon^{\mathsf{P}J}\colon \mathsf{F}J \to \mathsf{P}J$ send an (ultra)filter on a finite set to the element/subset it is principal at.

Let

$$\iota_X(\mathcal{F}) = \{\mathcal{U} \in \mathsf{U}X \mid \mathcal{F} \subseteq \mathcal{U}\} \tag{6.4}$$

for a set X and $\mathcal{F} \in \mathsf{F}X$. This gives a well-known section to π_X. Of course, $(\pi \cdot \iota)_X(\mathcal{F}) = \bigcap \iota_X(\mathcal{F}) \supseteq \mathcal{F}$. If $\mathcal{F} = \mathsf{P}X$, then the reverse inclusion is trivial. Otherwise, there is some $A \subseteq X$ not in $\mathcal{F}$, and so $\downarrow\!A$ is an ideal disjoint from $\mathcal{F}$. By the Ultrafilter Lemma, there is an ultrafilter $\mathcal{U}$ on X containing $\mathcal{F}$ disjoint from $\downarrow\!A$, and hence not containing A. This proves $\bigcap \iota_X(\mathcal{F}) \subseteq \mathcal{F}$.

The section ι_X is natural in X. Indeed, if $f\colon X \to Y$ is a function, $\mathcal{F} \in \mathsf{F}X$ and $\mathcal{U} \in \mathsf{U}Y$, then

$$\mathcal{U} \in (\iota_Y \cdot \mathsf{F}f)(\mathcal{F}) \iff \forall B \subseteq Y,\ (f^{-1}B \in \mathcal{F} \implies B \in \mathcal{U}),$$

and

$$\mathcal{U} \in (\mathsf{PU}f \cdot \iota_X)(\mathcal{F}) \iff \exists \mathcal{V} \in \mathsf{U}X,\ \mathcal{F} \subseteq \mathcal{V} \text{ and } \mathcal{U} = \mathsf{U}f(\mathcal{V}).$$

The second line certainly implies the first one. If $\mathcal{F} = \mathsf{P}X$ then the equivalence is clear, since no $\mathcal{U}$ satisfies either line. Now assume $\mathcal{F}$ is a proper filter. For $\mathcal{U}$ satisfying the first line, let

$$\mathcal{I} = \{A \subseteq X \mid \exists B \subseteq Y,\ B \notin \mathcal{U} \text{ and } A \subseteq f^{-1}B\}.$$

We claim that $\mathcal{I}$ is an ideal. It is certainly downwards closed and contains $\varnothing$. If $A_1 \subseteq f^{-1}B_1$ and $A_2 \subseteq f^{-1}B_2$ for some $B_1, B_2 \notin \mathcal{U}$, then $B_1 \cup B_2 \notin \mathcal{U}$ since the complement of an ultrafilter is an ideal, and $A_1 \cup A_2 \subseteq f^{-1}(B_1 \cup B_2)$. Moreover, $\mathcal{I}$ is disjoint from $\mathcal{F}$, for if $A \subseteq f^{-1}B$ with $B \notin \mathcal{U}$, then $f^{-1}B \notin \mathcal{F}$ by our assumption on $\mathcal{U}$, and thus $A \notin \mathcal{F}$. By the Ultrafilter Lemma, there is $\mathcal{V} \in \mathsf{U}X$ containing $\mathcal{F}$ and disjoint from $\mathcal{I}$. This disjointness means that $\mathsf{U}f(\mathcal{V})$ contains no $B \notin \mathcal{U}$, and so $\mathsf{U}f(\mathcal{V}) \subseteq \mathcal{U}$. Since ultrafilters are maximal with respect to inclusion, we must have $\mathsf{U}f(\mathcal{V}) = \mathcal{U}$. Hence $\mathcal{U}$ satisfies the second line above, concluding the proof of the naturality of ι.

It remains to check that ι is compatible with ϵ^J and $\epsilon^{\mathsf{P}J}$. If X is finite, then $\mathcal{F} = \uparrow\!A$ for some $A \subseteq X$ and $\epsilon_X^{\mathsf{P}J}(\mathcal{F}) = A$. Similarly, $\iota J_X(\mathcal{F}) = \{\uparrow\!\{a\} \mid a \in A\}$ which is sent by $\mathsf{P}\epsilon^J$ to A, as needed. □

The remaining ingredient which will allow us to apply Theorem 5.4.4 is the compatibility condition of the section ι with μ^{U}. Following the notation there, we let $\tau\colon \mathsf{U} \to \mathsf{F}$ be the obvious inclusion.

Lemma 6.1.17. *The section $\iota\colon \mathsf{F} \to \mathsf{PU}$ of the previous proposition satisfies the condition of Theorem 5.4.4(ii), i.e. the following diagram commutes*

$$\begin{tikzcd}
\mathsf{FU} \arrow[rr, "\iota\mathsf{U}"] \arrow[d, "\mathsf{F}\tau"'] & & \mathsf{PU}^2 \arrow[d, "\mathsf{P}\mu^{\mathsf{U}}"] \\
\mathsf{F}^2 \arrow[r, "\mu^{\mathsf{F}}"'] & \mathsf{F} \arrow[r, "\iota"'] & \mathsf{PU}.
\end{tikzcd}$$

Proof. Let X be a set and $\mathfrak{F} \in \mathsf{FU}X$. If $\mathfrak{F} = \mathsf{PU}X$ then both composites map it to $\varnothing$, so we assume $\mathfrak{F}$ is proper. The top composite sends $\mathfrak{F}$ to

$$\mathfrak{A} = \left\{ \{A \subseteq X \mid A^\sharp \cap \mathsf{U}X \in \mathfrak{U}\} \,\middle|\, \mathfrak{U} \in \mathsf{U}^2X \text{ such that } \mathfrak{F} \subseteq \mathfrak{U} \right\},$$

while the bottom composite produces

$$\mathfrak{B} = \{\mathcal{U} \in \mathsf{U}X \mid \forall A \subseteq X,\ A^\sharp \cap \mathsf{U}X \in \mathfrak{F} \implies A \in \mathcal{U}\}.$$

That $\mathfrak{A} \subseteq \mathfrak{B}$ is immediate from their definition.

For the other direction, let $\mathcal{U} \in \mathfrak{B}$ and define

$$\mathfrak{I} = \{B \subseteq \mathsf{U}X \mid \exists A \subseteq X,\ A \notin \mathcal{U} \text{ and } B \subseteq A^\sharp \cap \mathsf{U}X\}.$$

This is an ideal on $\mathsf{U}X$. Indeed, it is downwards closed and contains $\varnothing$. If $B_1 \subseteq A_1^\sharp \cap \mathsf{U}X$ and $B_2 \subseteq A_2^\sharp \cap \mathsf{U}X$ with $A_1, A_2 \notin \mathcal{U}$, then $A_1 \cup A_2 \notin \mathcal{U}$ since the complement of an ultrafilter is an ideal. Moreover,

$$B_1 \cup B_2 \subseteq (A_1^\sharp \cup A_2^\sharp) \cap \mathsf{U}X \subseteq (A_1 \cup A_2)^\sharp \cap \mathsf{U}X,$$

so $B_1 \cup B_2 \in \mathfrak{I}$.

Note that $\mathfrak{F}$ and $\mathfrak{I}$ are disjoint, since $A \notin \mathcal{U}$ implies that $A^\sharp \cap \mathsf{U}X \notin \mathfrak{F}$ because $\mathcal{U} \in \mathfrak{B}$. Thus, by the Ultrafilter Lemma, there exists $\mathfrak{U} \in \mathsf{U}^2X$ containing $\mathfrak{F}$ and disjoint from $\mathfrak{I}$. This disjointness means that if $A \notin \mathcal{U}$ then $A^\sharp \cap \mathsf{U}X \notin \mathfrak{U}$, so $\mu_X^{\mathsf{U}}(\mathfrak{U}) \subseteq \mathcal{U}$. Equality then follows from maximality. ☐

Theorem 6.1.18. *The filter monad $\mathsf{F} \cong J_\sharp\overline{\mathsf{P}}$ is the weak composite of a π-weak distributive law $\delta\colon \mathsf{UP} \to \mathsf{PU}$ given by*

$$\delta_X(\mathfrak{U}) = \left\{\mathcal{U} \in \mathsf{U}X \,\middle|\, \forall \mathfrak{A} \in \mathfrak{U},\ \bigcup \mathfrak{A} \in \mathcal{U}\right\}$$

for a set X and $\mathfrak{U} \in \mathsf{UP}X$.

Proof. As promised in Example 5.4.5, this is a direct application of Theorem 5.4.4, using Proposition 6.1.16 and the previous lemma. The formula for δ is obtained as the composite

$$\mathsf{UP} \xrightarrow{\tau\sigma} \mathsf{F}^2 \xrightarrow{\mu^{\mathsf{F}}} \mathsf{F} \xrightarrow{\iota} \mathsf{PU},$$

where we have used the notation of Section 5.4. Explicitly, ι is as in (6.4), τ is the inclusion and σ is as in Lemma 6.1.12, i.e. it sends a subset to the principal filter at it. ☐

One readily checks that δ in the previous theorem agrees with the weak distributive law of Garner [28, §5.1]. As explained there, it induces a weak lift of P to CHaus, which is given by the Vietoris monad. This monad sends a compact Hausdorff space X to the set of all its closed subsets, equipped with a suitable compact Hausdorff topology. Its category of algebras necessarily coincides with that of the filter monad, which is the category of continuous lattices, as we saw in Example 2.4.20. The facts that a continuous lattice is a special kind of complete lattice, and that it has a canonical compact Hausdorff topology are then automatic from the pushforward construction, by taking $T = \mathsf{P}$ in (6.2).

We conclude this section with the negative result promised after Proposition 5.1.8. Recall that P^{f} denotes the finite powerset monad (Example 2.4.17).

Proposition 6.1.19. *There is no distributive law* $\delta \colon \mathsf{U}\mathsf{P}^{\mathsf{f}} \to \mathsf{P}^{\mathsf{f}}\mathsf{U}$.

Proof. Note that the restriction of P^{f} along J is just $\overline{\mathsf{P}}$. Let $\nu \colon \mathsf{P}^{\mathsf{f}}\mathsf{U} \to \mathsf{F}$ be the natural transformation of Lemma 5.1.1. It sends a finite set of ultrafilters to their intersection. We show that ν is monic.

Let $\mathfrak{A}, \mathfrak{B} \in \mathsf{P}^{\mathsf{f}}\mathsf{U}X$ for a set X, and suppose $\mathcal{U} \in \mathfrak{A} \setminus \mathfrak{B}$. For each $\mathcal{V} \in \mathfrak{B}$, there is some $A_{\mathcal{V}} \in \mathcal{V} \setminus \mathcal{U}$. Since $\mathfrak{B}$ is finite and the complement of $\mathcal{U}$ is an ideal, $A = \bigcup_{\mathcal{V}\in\mathfrak{B}} A_{\mathcal{V}}$ is not in $\mathcal{U}$ but it is in every $\mathcal{V} \in \mathfrak{B}$. Thus, $A \in \nu(\mathfrak{B})$ but not in $\nu(\mathfrak{A})$.

By Proposition 5.1.8, it suffices to show that the monad structure of F does not restrict along ν. It is easy to see that the unit does restrict. However, we show that $\mu^{\mathsf{F}} \cdot \nu\nu$ does not factor through ν.

The image of ν consists exactly of those filters which are the intersection of a (uniquely determined) finite collection of ultrafilters. Indeed, if $\mathcal{F}$ is the intersection of both $\mathfrak{A} \in \mathsf{P}^{\mathsf{f}}\mathsf{U}X$ and $\mathfrak{B} \in \mathsf{P}\mathsf{U}X$, then it is also the intersection of any finite subcollection of $\mathfrak{A} \cup \mathfrak{B}$ containing $\mathfrak{A}$. Since ν is monic, $\mathfrak{A}$ must be the only such subcollection, so $\mathfrak{A} = \mathfrak{B}$.

Let X be an infinite set. We construct $\mathfrak{U} \in \mathsf{U}\mathsf{P}^{\mathsf{f}}\mathsf{U}X$ such that $(\mu^{\mathsf{F}} \cdot \nu\nu)_X(\{\mathfrak{U}\})$ is contained in infinitely may ultrafilters, and hence not in the image of ν. Let $\mathfrak{F}$ be the filter on $\mathsf{P}^{\mathsf{f}}\mathsf{U}X$ generated by the sets

$$\widehat{A} = \{\mathfrak{A} \in \mathsf{P}^{\mathsf{f}}\mathsf{U}X \mid \exists \mathcal{U} \in \mathfrak{A},\ A \in \mathcal{U}\}$$

for each nonempty $A \subseteq X$. This is a proper filter, because for any nonempty $A_1, \ldots, A_n \subseteq X$, each A_i is certainly contained in some $\mathcal{U}_i \in \mathsf{U}X$ and then $\{\mathcal{U}_1, \ldots, \mathcal{U}_n\} \in \widehat{A_1} \cap \cdots \cap \widehat{A_n}$. By the Ultrafilter Lemma, there is some ultrafilter $\mathfrak{U}$ containing $\mathfrak{F}$.

We claim that $(\mu^{\mathsf{F}} \cdot \nu\nu)_X(\{\mathfrak{U}\}) = \{A \subseteq X \mid \nu_X^{-1}(A^\sharp) \in \mathfrak{U}\}$ is the trivial filter, which is of course contained in infinitely many ultrafilters. Note that

$$\nu_X^{-1}(A^\sharp) = \{\mathfrak{A} \in \mathsf{P}^{\mathsf{f}}\mathsf{U}X \mid \forall \mathcal{U} \in \mathfrak{A},\ A \in \mathcal{U}\} = \left(\widehat{A^{\mathsf{C}}}\right)^{\mathsf{C}}.$$

If $A \subsetneq X$, then $A^{\mathsf{C}} \neq \varnothing$ and $\widehat{A^{\mathsf{C}}} \in \mathfrak{U}$ by construction. It follows that $\nu_X^{-1}(A^\sharp) \notin \mathfrak{U}$, and hence $A \notin (\mu^{\mathsf{F}} \cdot \nu\nu)_X(\{\mathfrak{U}\})$. □

The fact that there is no such distributive law means, among other things, that it is impossible to lift the finite powerset monad P^{f} along the underlying set functor $\mathsf{CHaus} \to \mathsf{Set}$.

There are analogous results if we take, instead of P and P^{f}, their respective submonads P_+ and $\mathsf{P}^{\mathsf{f}}_+$ which exclude the empty subset. They both restrict to $\overline{\mathsf{P}_+}$, and $J_\sharp\overline{\mathsf{P}_+}$ is isomorphic to the proper filter monad F_+. The proofs go through almost unchanged, giving:

Theorem 6.1.20. *The proper filter monad $\mathsf{F}_+ \cong J_\sharp\overline{\mathsf{P}_+}$ is the weak composite of a π-weak distributive law $\delta\colon \mathsf{UP}_+ \to \mathsf{P}_+\mathsf{U}$.*

Proposition 6.1.21. *There is no distributive law $\delta\colon \mathsf{UP}^{\mathsf{f}}_+ \to \mathsf{P}^{\mathsf{f}}_+\mathsf{U}$.*

6.1.2 The rank of pushforwards

It is common for codensity monads to not be finitary. For example, Di Liberti shows in [20, Thm. 4.4] that if $\mathcal{A}$ is a locally finitely presentable category, and the codensity monad T of the inclusion $\mathcal{A}_{\mathsf{fp}} \hookrightarrow \mathcal{A}$ is finitary, then T must be the identity. In this last subsection we show that, at least when taken along $J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$, pushforward monads have a similar tendency. In fact, except for those of the monads in Examples 6.1.1 and 6.1.2, i.e. the two inconsistent monads, pushforwards along J never have rank.

This will be an easy consequence of the fact that, for any regular cardinal κ, any subfunctor of a κ-accessible endofunctor of Set is κ-accessible. This can be proved from the fact that Set is strictly locally finitely presentable in the sense of Adámek, Milius, Sousa and Wißmann [2, Def. 3.9], and that, between such categories, κ-accessible functors coincide with κ-bounded ones (Theorem 4.11 in their article). However, we give an elementary direct proof here.

Proposition 6.1.22. *Let κ be a regular cardinal, and $T\colon \mathsf{Set} \to \mathsf{Set}$ be a κ-accessible functor. Any subfunctor of T is κ-accessible.*

Proof. We will use the fact that a functor out of Set is κ-accessible iff, for all sets X, it preserves the colimit of the canonical diagram $D_X\colon \mathsf{Sub}_\kappa(X) \to \mathsf{Set}$, where $\mathsf{Sub}_\kappa(X)$ is the poset of subobjects of X of cardinality less than κ. A proof of this (in the finitary case) can be found in [2, Cor. 2.7]. Recall that a colimit cocone for D_X is $(i\colon Y \hookrightarrow X)_{i\in\mathsf{Sub}_\kappa(X)}$.

Let $m\colon S \to T$ be a monic natural transformation, and consider the commutative square

$$\begin{tikzcd}
\operatorname{colim} SD_X \arrow[r] \arrow[d, hook, "\operatorname{colim} mD_x"'] & SX \arrow[d, hook, "m_X"] \\
\operatorname{colim} TD_X \arrow[r, "\cong"] & TX.
\end{tikzcd}$$

The injectivity of the function on the left follows the construction of κ-filtered colimits in Set, and the fact that mD_X is pointwise injective. Hence, the map on the top is injective, so it suffices to show that it is surjective.

This is trivial if $X = \varnothing$, so we assume otherwise. Let $s \in SX$. Then $m_X(s) = Ti(t)$ for some $i\colon Y \hookrightarrow X$ in $\mathsf{Sub}_\kappa(X)$ and $t \in TY$. Without loss of generality, we may assume that Y is nonempty, so that i has a retraction, say $ri = 1_Y$. Then

$$\begin{aligned} m_X \cdot Si \cdot Sr(s) &= Ti \cdot Tr \cdot m_X(s) && \text{by naturality} \\ &= Ti \cdot Tr \cdot Ti(t) \\ &= Ti(t) && \text{since } ri = 1_Y \\ &= m_X(s), \end{aligned}$$

and hence $Si \cdot Sr(s) = s$. This shows that s is in the image of $\operatorname{colim} SD_X \to SX$, and, since s was arbitrary, that this map is surjective. □

Remark 6.1.23. This proposition is crucially limited to functors whose domain is Set. For example, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in CRing, and since the Yoneda embedding $\mathsf{CRing}^{\mathrm{op}} \hookrightarrow [\mathsf{CRing}, \mathsf{Set}]$ preserves limits, it gives a monomorphism $\mathsf{CRing}(\mathbb{Q}, -) \to \mathsf{CRing}(\mathbb{Z}, -)$. However, as rings, $\mathbb{Z}$ is finitely presentable, while $\mathbb{Q}$ is not even finitely generated, so $\mathsf{CRing}(\mathbb{Z}, -)$ is $\aleph_0$-accessible, but $\mathsf{CRing}(\mathbb{Q}, -)$ is not.

Recall that a monad T on Set or FinSet is called consistent if η^T is monic.

Theorem 6.1.24. *Let T be a consistent monad on* FinSet*. Then $J_\sharp T$ has no rank.*

Proof. Since FinSet and Set have kernel pairs, monomorphisms in $\mathsf{Mnd}_{\mathsf{FinSet}}$ and $\mathsf{Mnd}_{\mathsf{Set}}$ are exactly those whose underlying natural transformation is monic (Remark 4.2.5), which coincide with those which are pointwise monic. As J preserves monomorphisms, both $J_\circ\colon [\mathsf{FinSet}, \mathsf{FinSet}] \to [\mathsf{FinSet}, \mathsf{Set}]$ and Ran_J (being a right adjoint) preserve monomorphisms, and hence so does $J_\sharp$. It follows that $J_\sharp \eta^T\colon \mathsf{U} \to J_\sharp T$ is a monic natural transformation. Since the ultrafilter monad does not have rank, Proposition 6.1.22 then implies that neither does $J_\sharp T$.

That the ultrafilter monad does not have rank is stated in several sources, e.g. in Borceux's book [15, Vol. II, p. 235], but seemingly always without proof. Here is the sketch of one: for a regular cardinal κ and a set X of cardinality at least κ, the collection of subsets of X whose complement has cardinality less than κ is a proper filter. By the Ultrafilter Lemma, it is contained in some ultrafilter $\mathcal{U}$. One checks that $\mathcal{U}$ is not in the image of $\mathsf{U}i$ for any $i \in \mathsf{Sub}_\kappa(X)$. It follows that the map $\operatorname{colim} \mathsf{U}D_X \to \mathsf{U}X$ in the proof of Proposition 6.1.22 is not surjective, and hence that U is not κ-accessible. □

6.2 Pushforwards from FinFam to Fam

Let $\mathcal{E}$ be a category with coproducts and cofiltered limits and $\mathsf{FinFam}(\mathcal{E})$ and $\mathsf{Fam}(\mathcal{E})$ denote its free completion under finite and small coproducts, respectively. Leinster [45, Thm. 8.5] showed that the codensity monad of the inclusion $\underline{J}\colon \mathsf{FinFam}(\mathcal{E}) \hookrightarrow \mathsf{Fam}(\mathcal{E})$ is the ultracoproduct monad. When $\mathcal{E} = (\mathsf{Set}^T)^{\mathrm{op}}$ for some finitary monad T, this sends an A-indexed family of T-algebras $(X_a)_{a\in A}$ to the $\mathsf{U}A$-indexed family of their ultraproducts $(\prod_{\mathcal{U}} X_a)_{\mathcal{U}\in\mathsf{U}A}$, in the sense of model theory. In this section, we compute the pushforward of a lift of the powerset monad $\overline{\mathsf{P}}$ to $\mathsf{FinFam}(\mathcal{E})$ along $\underline{J}$, and show that it is the reduced coproduct monad. The dual reduced products are again a familiar construction in model theory, as defined by Frayne, Morel and Scott in [27].

The author is grateful to Richard Garner for suggesting the study of pushforwards along $\underline{J}$ and helping to shape some of the results in this section.

The category $\mathsf{Fam}(\mathcal{E})$ is perhaps best introduced as the Grothendieck construction $\mathbf{E}(\mathcal{E}^{-})$ of the functor $\mathcal{E}^{-}\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{CAT}$ which sends a set A to $\mathcal{E}^{A} \cong \mathsf{CAT}(A,\mathcal{E})$. Explicitly, $\mathsf{Fam}(\mathcal{E})$ is the category where

- objects are pairs $(A,\underline{x})$ where A is a set and $\underline{x} = (x_a)_{a\in A}$ is an A-indexed family of objects of $\mathcal{E}$;

- morphisms $(A,\underline{x}) \to (B,\underline{y})$ are pairs $(f,\underline{\varphi})$ where $f\colon A \to B$ is a function and $\underline{\varphi} = (\varphi_a)_{a\in A}$ is an A-indexed family of morphisms $\varphi_a\colon x_a \to y_{f(b)}$ in $\mathcal{E}$.

The identity on $(A,\underline{x})$ is $(1_A,(1_{x_a})_{a\in A})$. The composite of

$$(A,\underline{x}) \xrightarrow{(f,\underline{\varphi})} (B,\underline{y}) \xrightarrow{(g,\underline{\psi})} (C,\underline{z})$$

is $(gf,(\psi_{f(a)}\varphi_a)_{a\in A})$. As the Grothendieck construction of $\mathcal{E}^{-}$, the category $\mathsf{Fam}(\mathcal{E})$ comes equipped with a (Grothendieck) fibration $P\colon \mathsf{Fam}(\mathcal{E}) \to \mathsf{Set}$.

The category $\mathsf{FinFam}(\mathcal{E})$ is easily recovered as the Grothendieck construction of $\mathcal{E}^{J-}\colon \mathsf{FinSet}^{\mathrm{op}} \to \mathsf{CAT}$. Its objects and morphisms are the same as those of $\mathsf{Fam}(\mathcal{E})$ except that we insist that the indexing sets are finite. It comes equipped with a fibration $Q\colon \mathsf{FinFam}(\mathcal{E}) \to \mathsf{FinSet}$ fitting into a commutative diagram

$$\begin{tikzcd}
\mathsf{FinFam}(\mathcal{E}) \arrow[r, "\underline{J}"] \arrow[d, "Q"'] & \mathsf{Fam}(\mathcal{E}) \arrow[d, "P"] \\
\mathsf{FinSet} \arrow[r, "J"'] & \mathsf{Set},
\end{tikzcd}$$

which is easily seen to be a pullback in CAT.

Yet another way to view $\mathsf{Fam}(\mathcal{E})$ is as the lax slice category $\mathsf{Set}/\!\!/\mathcal{E}$ in CAT, whose objects are functors $\Gamma^{A}\colon A \to \mathcal{E}$ from a discrete category A and whose morphisms from Γ^{A} to $\Gamma^{B}\colon B \to \mathcal{E}$ are pairs (f,φ) where $f\colon A \to B$ is a function and $\varphi\colon \Gamma^{A} \to \Gamma^{B} f$ is a natural transformation.

Leinster [45, Thm. 8.5] showed that $\underline{J}_{\sharp}1$ is the ultracoproduct monad $\underline{\mathsf{U}}$, which lifts the ultrafilter monad U along P. For $(A,\underline{x}) \in \mathsf{Fam}(\mathcal{E})$,

$$\underline{\mathsf{U}}(A,\underline{x}) = \left(\mathsf{U}A, \left(\lim_{S\in\mathcal{U}} \sum_{a\in S} x_a\right)_{\mathcal{U}\in\mathsf{U}A}\right),$$

where we view $\mathcal{U} \in \mathsf{U}\mathcal{A}$ as a poset under subset inclusion.

When $\mathcal{E} = (\mathsf{Set}^T)^{\mathrm{op}}$, for some finitary monad T on Set, this becomes

$$\underline{\mathsf{U}}(A, \underline{X}) = \left(\mathsf{U}A, (\textstyle\prod_{\mathcal{U}} \underline{X})_{\mathcal{U} \in \mathsf{U}A}\right),$$

where $\prod_{\mathcal{U}} \underline{X}$ is the ultraproduct of the A-indexed family of T-algebras $\underline{X}$ over the ultrafilter $\mathcal{U}$. Explicitly, it is the quotient of $\prod_{a \in A} X_a$ by the equivalence relation which identifies $(x_a)_{a \in A}$ and $(y_a)_{a \in A}$ whenever the set $\{a \in A \mid x_a = y_a\}$ is in $\mathcal{U}$. This construction is of great importance in model theory thanks to Łoś's theorem, which states that a first-order formula holds in $\prod_{\mathcal{U}} \underline{X}$ iff the set of those i such that the formula holds in X_i is in $\mathcal{U}$. Kennison [36, Thm. 1.4] showed that the category of $\underline{\mathsf{U}}$ algebras is isomorphic to the category T-Shf, whose objects are compact Hausdorff spaces equipped with a sheaf of T-algebras and whose morphisms are maps of sheaves which induce T-algebra homomorphisms on each stalk.

The following result, given by Leinster in [45, Lem. 8.2], is standard. It will allow us to compute certain pushforwards along $\underline{J}$. For a functor $\Sigma\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{CAT}$ and $f\colon a \to a'$ in $\mathcal{A}$ we write $f^*\colon \Sigma a' \to \Sigma a$ for the image of f under Σ.

Lemma 6.2.1. *Let $\mathcal{I}$ be a category, and let $\mathcal{A}$ be a category with limits of shape $\mathcal{I}$. Let $\Sigma\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{CAT}$ be a functor such that Σa has limits of shape $\mathcal{I}$ for each $a \in \mathcal{A}$ which are preserved by f^* for all $f\colon a' \to a$ in $\mathcal{A}$. Then the Grothendieck construction $\mathbf{E}(\Sigma)$ has limits of shape $\mathcal{I}$ and they are preserved by the projection $P\colon \mathbf{E}(\Sigma) \to \mathcal{A}$.*

Explicitly, if $D\colon \mathcal{I} \to \mathbf{E}(\Sigma)$ is a functor with $Di = (a_i, x_i)$ for $i \in \mathcal{I}$, then

$$\lim_{i \in \mathcal{I}} Di = \left(\lim_{i \in \mathcal{I}} a_i, \lim_{i \in \mathcal{I}} \pi_i^* x_i\right),$$

where $\pi_i\colon \lim_i a_i \to a_i$ are the legs of a limit cone in $\mathcal{A}$ for PD.

Recall from Proposition 4.3.2 that a functor $G\colon \mathcal{A} \to \mathcal{B}$ is representably small iff $b{\downarrow}G$ is cofinally small for each $b \in \mathcal{B}$. The next proposition is the analogue of Leinster's [45, Prop. 8.3] for pushforward monads. As we did in Section 6.1, if T is a monad on $\mathcal{A}$ whose pointwise pushforward along G exists and $f\colon b \to Ga$ is a morphism in $\mathcal{B}$, we write $\langle f \rangle\colon G_\sharp T(b) \to GTa$ for the leg of the limit cone of (4.6) at $f \in b{\downarrow}G$.

First, we need to introduce some notation. Let $\Sigma\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{CAT}$ be a functor, inducing the fibration $P\colon \mathbf{E}(\Sigma) \to \mathcal{A}$. Let T be an endofunctor on $\mathcal{A}$ and $\underline{T}$ be an endofunctor on $\mathbf{E}(\Sigma)$ which lifts T along P. For each $(a, \underline{x}) \in \mathbf{E}(\Sigma)$, we have

$$\underline{T}(a, \underline{x}) = (Ta, \underline{T}_a \underline{x})$$

for some $\underline{T}_a \underline{x} \in \Sigma Ta$. Note that $\underline{T}_a$ becomes a functor $\Sigma a \to \Sigma Ta$, since $\underline{T}(1_a, \varphi) = (1_{Ta}, \underline{T}_a \varphi)$ for some morphism $\underline{T}_a \varphi$ in ΣTa.

Proposition 6.2.2. *Let $G\colon \mathcal{A} \to \mathcal{B}$ and $\Sigma\colon \mathcal{B}^{\mathrm{op}} \to \mathsf{CAT}$ be functors. Suppose G is representably small, $\mathcal{B}$ is complete, Σb is complete for all $b \in \mathcal{B}$, and f^* has a* *left adjoint $f_!$ for every morphism f in $\mathcal{B}$. Consider the pullback diagram in* CAT

$$\begin{tikzcd}
\mathbf{E}(\Sigma G^{\mathrm{op}}) \arrow[r, "\underline{G}"] \arrow[d, "Q"'] & \mathbf{E}(\Sigma) \arrow[d, "P"] \\
\mathcal{A} \arrow[r, "G"'] & \mathcal{B}.
\end{tikzcd}$$

Let T be a monad on $\mathcal{A}$ and $\underline{T}$ be a monad on $\mathbf{E}(\Sigma G^{\mathrm{op}})$ which lifts T along Q. Then the pointwise pushforward $\underline{G_\sharp T}$ exists, it is given by

$$\underline{G_\sharp T}(b, \underline{x}) = \left(G_\sharp T(b), \lim_{\substack{f\colon b \to Ga \\ \in b\downarrow G}} \langle f \rangle^* \underline{T}_a(f_! \underline{x}) \right),$$

for $(b, \underline{x}) \in \mathbf{E}(\Sigma)$, and it is a lift of $G_\sharp T$ along P.

Proof. Using the pointwise formula for the pushforward in (4.6), for $(b, \underline{x}) \in \mathbf{E}(\Sigma)$ we have

$$\underline{G_\sharp T}(b, \underline{x}) = \lim \left((b, \underline{x})\downarrow\underline{G} \xrightarrow{\Pi_{(b,\underline{x})}} \mathbf{E}(\Sigma G^{\mathrm{op}}) \xrightarrow{\underline{T}} \mathbf{E}(\Sigma G^{\mathrm{op}}) \xrightarrow{\underline{G}} \mathbf{E}(\Sigma) \right)$$

where $\Pi_{(b,\underline{x})}$ is the projection.

An object of $(b, \underline{x})\downarrow\underline{G}$ consists of an object $a \in \mathcal{A}$, an object $\underline{y} \in \Sigma(Ga)$, a morphism $g\colon b \to Ga$ in $\mathcal{B}$, and a morphism $\varphi\colon \underline{x} \to f^*\underline{y}$ in Σb. By assumption, such a φ corresponds to a unique $\psi\colon f_!\underline{x} \to \underline{y}$ in $\Sigma(Ga)$. Hence, the limit above is equivalent to

$$\lim_{\substack{f\colon b \to Ga \\ \in b\downarrow G}} \lim_{\substack{\psi\colon f_!\underline{x} \to \underline{y} \\ \in f_!\underline{x}\downarrow\Sigma(Ga)}} \underline{GT}(a, \underline{y})$$

as soon as this limit exists. By assumption, $f_!\underline{x}\downarrow\Sigma(Ga)$ has an initial object, the identity on $f_!\underline{x}$, so the inside limit simplifies to $\underline{GT}(a, f_!\underline{x}) = (GTa, \underline{T}_a(f_!\underline{x}))$.

Let $S\colon \mathcal{I} \to b\downarrow G$ be a cofinal functor from a small category. This together with the completeness assumptions ensures that the limit exists by Lemma 6.2.1, and is given by

$$\left(G_\sharp T(b), \lim_{\substack{f\colon b \to Ga \\ \in b\downarrow G}} \langle f \rangle^* \underline{T}_a(f_! \underline{x}) \right).$$

That the monad structure of $\underline{G_\sharp T}$ lifts that of $G_\sharp T$ along P is immediate from its definition in terms of the universal property of limits and how they are computed in $\mathbf{E}(\Sigma)$. □

Remark 6.2.3. Note that instead of assuming that G is representably small and Σb is complete for all $b \in \mathcal{B}$, we could assume that, for each $b \in \mathcal{B}$, the category Σb has limits over all the shapes $b'\downarrow G$ for $b' \in \mathcal{B}$.

When $\mathcal{E}$ has coproducts and cofiltered limits, taking $\Sigma = \mathcal{E}^-\colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{CAT}$ and $G = J\colon \mathsf{FinSet} \hookrightarrow \mathsf{Set}$ satisfies the hypotheses of the previous remark. Given a function $f\colon A \to B$, the reindexing functor $f^*\colon \mathcal{E}^B \to \mathcal{E}^A$ has a left adjoint $f_!$ as soon as $\mathcal{E}$ has coproducts. Indeed,

$$\mathcal{E}^A(\underline{x}, f^*\underline{y}) \cong \prod_{a\in A} \mathcal{E}(x_a, y_{f(a)}) \cong \prod_{b\in B} \mathcal{E}(\textstyle\sum_{f(a)=b} x_a, y_b),$$

so $f_!\underline{x}$ has $\sum_{a\in f^{-1}\{b\}} x_a$ as its b-th component. Moreover, for every set A, the category $A{\downarrow}J$ is cofiltered, so that each $\mathcal{E}^B$ has limits of shape $A{\downarrow}J$ (computed componentwise).

We now introduce a lift $\underline{\mathsf{P}}$ of the powerset monad P along $P\colon \mathsf{Fam}(\mathcal{E}) \to \mathsf{Set}$. It will itself lift along $\underline{J}$ to produce a lift $\overline{\underline{\mathsf{P}}}$ of $\overline{\mathsf{P}}$ along $Q\colon \mathsf{FinFam}(\mathcal{E}) \to \mathsf{FinSet}$. For $(A, \underline{x}) \in \mathsf{Fam}(\mathcal{E})$, define

$$\underline{\mathsf{P}}(A, \underline{x}) = \Big(\mathsf{P}A, (\textstyle\sum_{a\in S} x_a)_{S\subseteq A}\Big)\,. \tag{6.5}$$

For a morphism $(f, \underline{\varphi})\colon (A, \underline{x}) \to (B, \underline{y})$ in $\mathsf{Fam}(\mathcal{E})$, we put

$$\underline{\mathsf{P}}(f, \underline{\varphi}) = \Big(\mathsf{P}f, (\varphi_S)_{S\subseteq A}\Big)$$

where $\varphi_S\colon \sum_{a\in S} x_a \to \sum_{a\in S} y_{f(a)}$ is the coproduct of the components of $\underline{\varphi}$ in S. It is straightforward to see that $\underline{\mathsf{P}}$ is a functor, which we endow with the following monad structure.

The unit $\eta^{\underline{\mathsf{P}}}$ at $(A, \underline{x})$ has η^{P}_A as its first component and the canonical isomorphism $x_a \to \sum_{a'\in\{a\}} x_{a'}$ for each $a \in A$ as its second component. The multiplication $\mu^{\underline{\mathsf{P}}}$ at $(A, \underline{x})$ has μ^{P}_A as its first component and, for $\mathfrak{A} \subseteq \mathsf{P}A$, the canonical isomorphism

$$\mu^{\underline{\mathsf{P}}}_{\mathfrak{A}}\colon \sum_{A\in\mathfrak{A}} \sum_{a\in A} x_a \longrightarrow \sum_{a\in\mu^{\mathsf{P}}_A(\mathfrak{A})} x_a$$

as its second component. Once again, it is easy to see that this defines a monad $\underline{\mathsf{P}}$ and that it lifts P along P. Just like P restricts along J to $\overline{\mathsf{P}}$, the monad $\underline{\mathsf{P}}$ restricts along $\underline{J}$ to the monad $\overline{\underline{\mathsf{P}}}$ on $\mathsf{FinFam}(\mathcal{E})$.

The $\underline{\mathsf{P}}$-algebras admit a simple description:

Proposition 6.2.4. *The category of $\underline{\mathsf{P}}$-algebras is isomorphic to the lax slice category $\mathsf{SupLat}/\!/\mathcal{E}$ in CAT, where $\mathsf{SupLat} = \mathsf{Set}^{\mathsf{P}}$ is the category of suplattices and supremum-preserving maps.*

Proof. Let $((A, \underline{x}), (\alpha, \underline{\alpha}))$ be a $\underline{\mathsf{P}}$-algebra. Since $\underline{\mathsf{P}}$ is a lift of P, it follows that (A, α) is a P-algebra, i.e. a suplattice. For each $S \subseteq A$, we have a morphism

$$\alpha_S\colon \sum_{a\in S} x_a \longrightarrow x_{\alpha(S)}$$

in $\mathcal{E}$. The suplattice A becomes a poset, and hence a category, by setting $a \le a'$ iff $\alpha(\{a, a'\}) = a'$. Whenever this is the case, we have a morphism

$$\Gamma(a \le a') = x_a \xrightarrow{\iota_a} \sum_{b\in\{a,a'\}} x_b \xrightarrow{\alpha_{\{a,a'\}}} x_{a'}$$

in $\mathcal{E}$, where ι_a is the coproduct inclusion. The unit axiom for the $\underline{\mathsf{P}}$-algebra ensures that $\Gamma(a \leq a)$ is the identity on x_a. If $a \leq a' \leq a'' \in A$, let $\mathfrak{S} = \{\{a, a'\}, \{a', a'''\}\}$ and consider the diagram

$$\begin{tikzcd}
x_a \arrow[r, "\iota_a"] \arrow[rrr, bend left, "\iota_a"] & \sum\limits_{b \in \{a,a'\}} x_b \arrow[r, "\iota_{\{a,a'\}}"] \arrow[d, "\alpha_{\{a,a'\}}"'] & \sum\limits_{S \in \mathfrak{S}} \sum\limits_{b \in S} x_b \arrow[r, "\mu_{\mathfrak{S}}"] \arrow[d, "\underline{\mathsf{P}}(\alpha, \underline{\alpha})_{\mathfrak{S}}"'] & \sum\limits_{b \in \{a,a',a''\}} x_b \arrow[d, "\alpha_{\{a,a',a''\}}"] \\
& x_{a'} \arrow[r, "\iota_{a'}"'] & \sum\limits_{b \in \{a',a''\}} x_b \arrow[r, "\alpha_{\{a',a''\}}"'] & x_{a''},
\end{tikzcd}$$

where the top face commutes by definition of μ, the left square by definition of $\underline{\mathsf{P}}$ and the right square by the $\underline{\mathsf{P}}$-algebra multiplication axiom. The bottom composite is $\Gamma(a' \leq a'') \cdot \Gamma(a \leq a')$. A similar diagram shows that the top composite is also equal to $\Gamma(a \leq a'')$, proving that Γ is a functor.

Conversely, suppose we have a suplattice (A, α) and a functor $\Gamma \colon A \to \mathcal{E}$. Let $\underline{x} = (\Gamma a)_{a \in A}$ and, for $S \subseteq A$, define

$$\alpha_S \colon \sum_{a \in S} x_a \longrightarrow a_{\alpha(S)}$$

to be the map induced by $\Gamma(a \leq \alpha(S))$ for $a \in S$. The $\underline{\mathsf{P}}$-algebra unit axiom amounts the fact that these maps are identities when S is a singleton. The multiplication axiom is similarly checked, using the fact that Γ is a functor and that maps out of a coproduct are determined by their components.

A morphism $(f, \underline{\varphi}) \colon ((A, \underline{x}), (\alpha, \underline{\alpha})) \to ((B, \underline{y}), (\beta, \underline{\beta}))$ of $\underline{\mathsf{P}}$-algebras amounts to a supremum-preserving map $f \colon A \to B$ together with morphisms $\varphi_a \colon x_a \to y_{f(a)}$ in $\mathcal{E}$ for each $a \in A$ such that

$$\begin{tikzcd}
\sum\limits_{a \in S} x_a \arrow[r, "\underline{\mathsf{P}}(f, \underline{\varphi})_S"] \arrow[d, "\alpha_S"'] & \sum\limits_{b \in f(S)} y_b \arrow[d, "\beta_{f(S)}"] \\
x_{\alpha(S)} \arrow[r, "\varphi_{\alpha(S)}"'] & y_{\beta(f(S))}
\end{tikzcd}$$

commutes for each $S \subseteq A$. Writing $\Gamma^A \colon A \to \mathcal{E}$ and $\Gamma^B \colon B \to \mathcal{E}$ for the corresponding functors, this gives a natural transformation $\Gamma^A \to \Gamma^B f$. Naturality follows from taking $S = \{a, a'\}$ for $a \leq a' \in A$ in the above diagram.

Conversely, given a supremum-preserving map $f \colon A \to B$ and a natural transformation $\varphi \colon \Gamma^A \to \Gamma^B f$, we get a morphism of $\underline{\mathsf{P}}$-algebras. That the diagram above commutes for each $S \subseteq A$ is checked by precomposing with each coproduct inclusion and using naturality. $\square$

We now compute the pushforward of the restricted monad $\overline{\underline{\mathsf{P}}}$ on $\mathsf{FinFam}(\mathcal{E})$ along $\underline{J} \colon \mathsf{FinFam}(\mathcal{E}) \to \mathsf{Fam}(\mathcal{E})$.

Theorem 6.2.5. *Let $\mathcal{E}$ be a category with coproducts and cofiltered limits. The pointwise pushforward $\underline{J_\sharp \overline{\mathsf{P}}}$ exists and is given by*

$$\underline{J_\sharp \overline{\mathsf{P}}}(A, \underline{x}) = \left(\mathsf{F}A, \left(\lim_{S \in \mathcal{F}} \sum_{a \in S} x_a\right)_{\mathcal{F} \in \mathsf{F}A}\right)$$

for $(A, \underline{x}) \in \mathsf{Fam}(\mathcal{E})$, where we view $\mathcal{F} \in \mathsf{F}A$ as a poset under subset inclusion.

Proof. We use the formula in Proposition 6.2.2, which is valid in this case by the discussion above. By Theorem 6.1.14, the first component of $\underline{J_\sharp \overline{\mathsf{P}}}(A, \underline{x})$ is exactly $\mathsf{F}A$. The second is given by

$$\begin{aligned}
\lim_{\substack{f\colon A \to B \\ \in A\downarrow J}} \langle f\rangle^* \underline{\overline{\mathsf{P}}_B}(f_! \underline{x}) &= \lim_{\substack{f\colon A \to B \\ \in A\downarrow J}} \langle f\rangle^* \underline{\overline{\mathsf{P}}_B} \left(\textstyle\sum_{f(a)=b} x_a\right)_{b \in B} \\
&= \lim_{\substack{f\colon A \to B \\ \in A\downarrow J}} \langle f\rangle^* \left(\sum_{b \in T} \sum_{f(a)=b} x_a\right)_{T \subseteq B} && \text{by (6.5)} \\
&= \lim_{\substack{f\colon A \to B \\ \in A\downarrow J}} \langle f\rangle^* \left(\sum_{a \in f^{-1}T} x_a\right)_{T \subseteq B} \\
&= \lim_{\substack{f\colon A \to B \\ \in A\downarrow J}} \left(\sum_{a \in f^{-1}\langle f\rangle(\mathcal{F})} x_a\right)_{\mathcal{F} \in \mathsf{F}A} \\
&= \left(\lim_{\substack{f\colon A \to B \\ \in A\downarrow J}} \sum_{a \in f^{-1}\langle f\rangle(\mathcal{F})} x_a\right)_{\mathcal{F} \in \mathsf{F}A},
\end{aligned}$$

since limits in $\mathcal{E}^{\mathsf{F}A}$ are computed componentwise. We have omitted the isomorphism $\psi\colon \mathsf{F} \to \underline{J_\sharp \mathsf{P}}$ of Theorem 6.1.14, so that $\langle f\rangle(\mathcal{F})$ is the smallest subset T of B such that $f^{-1}T \in \mathcal{F}$.

For $\mathcal{F} \in \mathsf{F}A$, let $D_{\mathcal{F}}\colon A\downarrow J \to \mathcal{E}$ be the diagram whose limit gives the $\mathcal{F}$-component above. If $f\colon A \to B \in A\downarrow J$ and $g\colon B \to C \in \mathsf{FinSet}$, then $\langle gf\rangle(\mathcal{F}) = g(\langle f\rangle(\mathcal{F}))$ by (6.1). Since $T \subseteq g^{-1}g(T)$ for any $T \subseteq B$, we then have

$$f^{-1}\langle f\rangle(\mathcal{F}) \subseteq (gf)^{-1}\langle gf\rangle(\mathcal{F})$$

with both sets in $\mathcal{F}$. Then $D_{\mathcal{F}}g$ is the map

$$\sum_{a \in f^{-1}\langle f\rangle(\mathcal{F})} x_a \longrightarrow \sum_{a \in (gf)^{-1}\langle gf\rangle(\mathcal{F})} x_a$$

induced by the coproduct inclusions.

Let $E_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{E}$ be the diagram in the statement, which sends $S \subseteq S' \in \mathcal{F}$ to the map $\sum_{a \in S} x_a \to \sum_{a \in S'} x_a$ induced by the coproduct inclusions. We have just shown that the assignment $f \mapsto f^{-1}\langle f\rangle(\mathcal{F})$ gives a functor $K\colon A\downarrow J \to \mathcal{F}$, and that $E_{\mathcal{F}}K = D_{\mathcal{F}}$. It suffices to prove that this functor is cofinal. This in turn follows easily from the fact that K is surjective on objects, $A\downarrow J$ is cofiltered and

$\mathcal{F}$ is a poset. $\square$

Being a lift of the filter monad F on Set along P, we denote $\underline{J}_\sharp\overline{\mathsf{P}}$ by $\underline{\mathsf{F}}$. By the general theory of Section 5.1, $\underline{\mathsf{F}}$ ought to be thought of as a combination of $\underline{J}_\sharp 1 \cong \underline{\mathsf{U}}$ and $\underline{\mathsf{P}}$. How exactly these two monads are combined does not seem to have a simple description. In particular, the π-weak distributive law of U over P yielding F does not to lift to a π-weak distributive law of $\underline{\mathsf{U}}$ over $\underline{\mathsf{P}}$ yielding $\underline{J}_\sharp\overline{\mathsf{P}}$. Already the section $\iota\colon \mathsf{F} \to \mathsf{PU}$ of Proposition 6.1.16 does not lift to $\mathsf{Fam}(\mathcal{E})$. A lift would require, for each $\mathcal{F} \in \mathsf{F}A$, a map

$$\lim_{S\in\mathcal{F}} \sum_{a\in S} x_a \longrightarrow \sum_{\substack{\mathcal{U}\in\mathsf{U}A \\ \mathcal{F}\subseteq\mathcal{U}}} \lim_{S\in\mathcal{U}} \sum_{a\in S} x_a,$$

which seems impossible to describe in this generality.

Setting $\mathcal{E} = (\mathsf{Set}^T)^{\mathrm{op}}$ for a finitary monad T, we have

$$\underline{\mathsf{F}}(A, \underline{X}) = \left(\mathsf{F}A, \left(\operatorname*{colim}_{S\in\mathcal{F}} \prod_{a\in S} X_a\right)_{\mathcal{F}\in\mathsf{F}A}\right).$$

For $\mathcal{F} \in \mathsf{F}A$, the colimit in the $\mathcal{F}$-component is the **reduced product** $\prod_{\mathcal{F}} \underline{X}$ of $\underline{X}$ over $\mathcal{F}$, as defined by Frayne, Morel and Scott in [27]. Much like the ultraproduct, this is the quotient of $\prod_{a\in S} X_a$ by the equivalence relation which identifies $(x_a)_{a\in A}$ and $(y_a)_{a\in A}$ whenever $\{a \in A \mid x_a = y_a\}$ is in $\mathcal{F}$.

Every $\underline{\mathsf{F}}$-algebra $(A, \underline{X})$ has underlying $\underline{\mathsf{P}}$- and $\underline{\mathsf{U}}$-algebra structures. The latter implies that A is a compact Hausdorff space with a sheaf of T-algebras. By Proposition 6.2.4, the former implies that A is a suplattice equipped with a functor $A^{\mathrm{op}} \to \mathsf{Set}^T$. Since $\underline{\mathsf{F}}$ lifts F, it also follows that A is in fact a continuous lattice. An explicit description of the $\underline{\mathsf{F}}$-algebras would involve specifying how these two structures interact. This is left as a subject for future work.

6.3 Codensity monads of the category of fields

In this section, we study the codensity monad K of the inclusion $\mathsf{Field} \hookrightarrow \mathsf{CRing}$ of the category of fields into the category of commutative unital rings (which we will simply refer to as rings within this section). Although we already saw in Example 4.3.23(i) that the category of K-algebras is the free product completion of Field using results of Diers's, this was proved independently by the author. This alternative derivation, unlike Diers's theory of multimonadic functors, is restricted to the categories in question. On the other hand, it is far more explicit.

The action of K on objects was described by Leinster [45, Ex. 5.1]. For a ring R we have

$$\mathsf{K}R \cong \prod_{\mathfrak{p}\in\operatorname{Spec} R} \operatorname{Frac}(R/\mathfrak{p}). \tag{6.6}$$

This can be seen from the limit formula (4.6) using the description of the category R/Field in Example 4.3.4(iv).

For $\mathfrak{p} \in \operatorname{Spec} R$, we will write $\pi_\mathfrak{p}$ for the projection $\mathsf{K}R \to \operatorname{Frac}(R/\mathfrak{p})$. The action of K on morphisms is as follows: for a ring homomorphism $f\colon R \to S$, $x \in \mathsf{K}R$ and $\mathfrak{q} \in \operatorname{Spec} R$, we have

$$\mathsf{K}f(x)_\mathfrak{q} = f_\mathfrak{q}(x_{f^{-1}\mathfrak{q}}), \tag{6.7}$$

where $f_\mathfrak{q}$ denotes the map of fields $\operatorname{Frac}(R/f^{-1}\mathfrak{q}) \to \operatorname{Frac}(S/\mathfrak{q})$ that f induces.

On the $\mathfrak{p}$-component, the unit η^K_R is the canonical map $R \to \operatorname{Frac}(R/\mathfrak{p})$. This unit embodies the philosophy of algebraic geometry: $\mathsf{K}R$ is a ring of dependent functions on $\operatorname{Spec} R$, and η^K_R realises elements of R as such functions. To understand the multiplication, we need the following elementary result about ideals in a product of fields.

Proposition 6.3.1. *Let I be a set, and k_i be a field for each i. The poset of ideals of $R = \prod_{i\in I} k_i$ is isomorphic to the poset of filters on I. All prime ideals are maximal, and they correspond to the ultrafilters. Moreover, for $S \subseteq I$, the quotient map from R to its quotient by the ideal corresponding to $\uparrow S$ is the projection*

$$\prod_{i\in I} k_i \longrightarrow \prod_{i\in S} k_i.$$

Proof. First, we introduce some notation. For $x \in R$, let $V(x) = \{i \in I \mid x_i = 0\}$. Given an ideal $\mathfrak{a}$ of R, we define

$$\mathcal{F}_\mathfrak{a} = \{V(x) \mid x \in \mathfrak{a}\}.$$

One easily checks that this is a filter on I. Conversely, a filter $\mathcal{F}$ on I gives the ideal

$$\mathfrak{a}_\mathcal{F} = \{x \in R \mid V(x) \in \mathcal{F}\}.$$

It is clear that these two processes are mutually inverse, and that they preserve inclusions.

If $\mathfrak{a}$ is a non-maximal proper ideal, then $\mathcal{F}_\mathfrak{a}$ is proper but not an ultrafilter, so there is some $S \subseteq I$ such that $S, I \setminus S \notin \mathcal{F}_\mathfrak{a}$. Taking $x, y \in R$ such that $V(x) = S$ and $V(y) = I \setminus S$ then gives $xy = 0 \in \mathfrak{a}$ and yet $x, y \notin \mathfrak{a}$, so $\mathfrak{a}$ is not prime.

For the statement about quotients, it suffices to note that, for $S \subseteq I$, the ideal $\mathfrak{a}_{\uparrow S}$ is precisely $\prod_{i\in I\setminus S} k_i \times \prod_{i\in S}\{0\} \subseteq R$. ☐

The multiplication $\mu^\mathsf{K}_R\colon \mathsf{K}^2 R \to \mathsf{K}R$ is then just the projection onto those factors corresponding to the principal ultrafilters:

$$\prod_{\mathfrak{q}\in\operatorname{Spec}\mathsf{K}R} \operatorname{Frac}(\mathsf{K}R/\mathfrak{q}) \longrightarrow \prod_{\mathfrak{p}\in\operatorname{Spec} R} \operatorname{Frac}(\mathsf{K}R/\mathfrak{a}_{\uparrow\{\mathfrak{p}\}}) \cong \prod_{\mathfrak{p}\in\operatorname{Spec} R} \operatorname{Frac}(R/\mathfrak{p})$$

Since $\mathsf{Field} \hookrightarrow \mathsf{CRing}$ is fully faithful, so is the comparison functor $\mathsf{Field} \hookrightarrow \mathsf{CRing}^\mathsf{K}$ by Remark 4.3.12. Thus every field is an example of a K-algebra. All the free K-algebras are products of fields by (6.6), and in fact this will be true for all K-algebras. To show this, we introduce the following notation: for a set $S \subseteq I$, and fields k_i for $i \in I$, let δ_S be the element of $\prod_{i\in I} k_i$ that is 1 in those components in S and 0 elsewhere.

Proposition 6.3.2. *Let $\xi\colon \mathsf{K}R \to R$ be a K-algebra structure map. Then $R \cong \prod_{i\in I} k_i$ for some set I and fields k_i for each $i \in I$, and ξ is isomorphic to the projection onto the components corresponding to the principal ultrafilters on I.*

Proof. Let $S \subseteq \operatorname{Spec}\mathsf{K}R$ be the set of those primes corresponding to nonprincipal ultrafilters. Our description of μ_R^{K} implies that $\mu_R^{\mathsf{K}}(\delta_S) = 0$, and then the multiplication axiom says that $\mathsf{K}\xi(\delta_S) \in \ker\xi$. By the description of K on maps in (6.7), $\mathsf{K}\xi(\delta_S) = \delta_T$, where $T \subseteq \operatorname{Spec}R$ is the set of those primes $\mathfrak{p}$ such that $\xi^{-1}\mathfrak{p} \in S$. Hence, $\delta_T \in \ker\xi$.

By the unit axiom, if $x \in R$ is nonzero, then $\eta_R^{\mathsf{K}}(x) \notin \ker\xi$, so $\eta_R^{\mathsf{K}}(x)$ is not a multiple of δ_T. Using the product formula (6.6), this means that $\eta_R^{\mathsf{K}}(x)$ is nonzero in at least one component outside of T. In other words, for every nonzero $x \in R$, there exists some $\mathfrak{p} \in \operatorname{Spec}R$ such that $x \notin \mathfrak{p}$ and $\xi^{-1}\mathfrak{p}$ corresponds to a principal ultrafilter. It follows that

$$\bigcap_{\mathfrak{p}\in(\operatorname{Spec}R)\setminus T} \mathfrak{p} = 0 \qquad \text{and} \qquad \ker\xi = \bigcap_{\mathfrak{p}\in(\operatorname{Spec}R)\setminus T} \xi^{-1}\mathfrak{p}.$$

Thus, $\ker\xi$ corresponds to an intersection of principal ultrafilters, which is itself a principal filter. Since $R \cong \mathsf{K}R/\ker\xi$, Proposition 6.3.1 gives that ξ is isomorphic to a projection onto a subproduct of $\mathsf{K}R$.

Now assume that $R = \prod_{i\in I} k_i$ for some set I and fields k_i for each $i \in I$. Then

$$\bigcap_{i\in I} \mathfrak{a}_{\uparrow\{i\}} = 0 \qquad \text{and} \qquad \ker\xi = \bigcap_{i\in I} \xi^{-1}\mathfrak{a}_{\uparrow\{i\}},$$

so it suffices to show that $\xi^{-1}\mathfrak{a}_{\uparrow\{i\}} = \mathfrak{a}_{\uparrow\{\uparrow\{i\}\}}$. Let $x \in R$ be such that $V(x) = \{i\}$, so $x \in \mathfrak{a}_{\uparrow\{i\}}$ and this is the only prime ideal of R containing x. Then $V(\eta_R^{\mathsf{K}}(x)) = \{\uparrow\{i\}\}$, and the only prime ideal of $\mathsf{K}R$ containing $\eta_R^{\mathsf{K}}(x)$ is $\mathfrak{a}_{\uparrow\{\uparrow\{i\}\}}$. Since $\xi\eta_R^{\mathsf{K}}(x) = x$ and $x \in \mathfrak{a}_{\uparrow\{i\}}$, this ideal must be $\xi^{-1}\mathfrak{a}_{\uparrow\{i\}}$. ☐

Since CRing has and U^{K} creates products, $\mathsf{CRing}^{\mathsf{K}}$ has products. Proposition 6.3.2 then implies that its objects are precisely the products of fields, each with a unique K-algebra structure. However, not every morphism of rings between two products of fields is a K-algebra map.

Proposition 6.3.3. *Let I and J be sets, and k_i and h_j be fields for each $i \in I$ and $j \in J$. For a ring homomorphism $f\colon \prod_{i\in I} k_i \to \prod_{j\in J} h_j$, the following are equivalent:*

(i) *f is a K-algebra map;*

(ii) *for each $j \in J$, the prime ideal $f^{-1}\mathfrak{a}_{\uparrow\{j\}}$ corresponds to a principal ultrafilter on I under the correspondence of Proposition 6.3.1;*

(iii) *there exist a function $f^*\colon J \to I$, and field homomorphisms $f_j\colon k_{f^*(j)} \to h_j$ for each $j \in J$, such that $\pi_j f = f_j \pi_{f^*(j)}$ for all $j \in J$.*

Proof. Throughout, let κ and λ be the unique algebra structure maps of $\prod_{i\in I} k_i$ and $\prod_{j\in J} h_j$, respectively.

(i) ⇒ (ii). Let $j \in J$ and $\mathfrak{p} = f^{-1}\mathfrak{a}_{\uparrow\{j\}}$. Consider $\delta_{\{\mathfrak{p}\}} \in \mathsf{K}(\prod_{i \in I} k_i)$. If $\mathfrak{p}$ is not principal, then $\kappa(\delta_{\{\mathfrak{p}\}}) = 0$ and, since f is a K-algebra map, $\lambda \cdot \mathsf{K}f(\delta_{\{\mathfrak{p}\}}) = 0$. This is a contradiction, because $\mathsf{K}f(\delta_{\{\mathfrak{p}\}})$ is 1 in the $\mathfrak{a}_{\uparrow\{j\}}$-component, and hence so is $\lambda \cdot \mathsf{K}f(\delta_{\{\mathfrak{p}\}})$ in the j-component.

(ii) ⇒ (iii). The function $f^* \colon J \to I$ sends $j \in J$ to the unique $i \in I$ such that $f^{-1}\mathfrak{a}_{\uparrow\{j\}} = \mathfrak{a}_{\uparrow\{i\}}$. The field homomorphism $f_j \colon k_{f^*(j)} \to h_j$ is the one induced by f on the quotients by $\mathfrak{a}_{\uparrow\{f^*(j)\}}$ and $\mathfrak{a}_{\uparrow\{j\}}$, which are $k_{f^*(j)}$ and h_j, respectively. The equations needed hold because these quotient maps are precisely the projections.

(iii) ⇒ (i). Let $x \in \mathsf{K}(\prod_{i \in I} k_i)$. The j-component of $f\kappa(x)$ is $f_j \pi_{f^*(j)} \kappa(x)$. To find the j-component of $\lambda \cdot \mathsf{K}f(x)$, we must first find $f^{-1}\mathfrak{a}_{\uparrow\{j\}}$. This is precisely the kernel of $\pi_j f = f_j \pi_{f^*(j)}$, which is $\ker \pi_{f^*(j)} = \mathfrak{a}_{\uparrow\{f^*(j)\}}$, since f_j is a field homomorphism, and hence injective. Then

$$\pi_j \lambda \cdot \mathsf{K}f(x) = f_j \pi_{\uparrow\{f^*(j)\}}(x) = f_j \pi_{f^*(j)} \kappa(x)$$

as needed. □

Condition (iii) is exactly what it means for f to be a morphism in the free product completion of Field.

Theorem 6.3.4. *The category* $\mathsf{Prod}(\mathsf{Field})$ *is monadic over* CRing*, and the corresponding monad is the codensity monad of the inclusion* $\mathsf{Field} \hookrightarrow \mathsf{CRing}$.

Proof. By Proposition 6.3.2, every object of $\mathsf{CRing}^{\mathsf{K}}$ is a product of fields, and each such product has a unique K-algebra structure. This establishes a bijection between the objects of $\mathsf{CRing}^{\mathsf{K}}$ and the objects of $\mathsf{Prod}(\mathsf{Field})$. Proposition 6.3.3 gives the bijection between the morphisms. □

Next, we examine the codensity monad of the forgetful functor $\mathsf{Field} \to \mathsf{Set}$. This forgetful functor is $U^{\mathsf{R}}J$, where R is the free ring monad on Set and $J \colon \mathsf{Field} \hookrightarrow \mathsf{CRing}$ and $U^{\mathsf{R}} \colon \mathsf{CRing} \to \mathsf{Set}$ are the forgetful functors. Since U^{R} is a right adjoint, Remark 4.1.8 gives

$$(U^{\mathsf{R}}J)_\sharp 1 \cong U^{\mathsf{R}}_\sharp(J_\sharp 1) = U^{\mathsf{R}}_\sharp \mathsf{K}.$$

By Lemma 2.2.3(iii), $U^{\mathsf{R}}_\sharp \mathsf{K} \cong U^{\mathsf{R}} \mathsf{K} F^{\mathsf{R}}$, which is the monad induced by the composite adjunction

$$\begin{tikzcd}
\mathsf{Set} \arrow[r, bend left, "F^{\mathsf{R}}"] \arrow[r, phantom, "\perp"] & \mathsf{CRing} \arrow[l, bend left, "U^{\mathsf{R}}"] \arrow[r, bend left, "F^{\mathsf{K}}"] \arrow[r, phantom, "\perp"] & \mathsf{Prod}(\mathsf{Field}) \arrow[l, bend left, "U^{\mathsf{K}}"]
\end{tikzcd}$$

The next proposition sets up the use of the Crude Monadicity Theorem.

Proposition 6.3.5. *The category* $\mathsf{Prod}(\mathsf{Field})$ *has and* $U^{\mathsf{R}}U^{\mathsf{K}}$ *preserves reflexive coequalisers.*

Proof. Let I and J be sets, k_i and h_j be fields for each $i \in I$ and $j \in J$, and $f, g \colon \prod_{i \in I} k_i \rightrightarrows \prod_{j \in J} h_j$ be morphisms in $\mathsf{Prod}(\mathsf{Field})$ with common section s.

Let us focus on a single $j \in J$. We have two field homomorphisms

$$f_j \colon k_{f^*(j)} \longrightarrow h_j \qquad \text{and} \qquad g_j \colon k_{g^*(j)} \longrightarrow h_j,$$

such that $f_j s_{f^*(j)} = 1 = g_j s_{g^*(j)}$. Since all field homomorphisms are injective, this implies that f_j and g_j are isomorphisms with inverses $s_{f^*(j)}$ and $s_{g^*(j)}$, respectively.

Now let

$$J' = \{j \in J \mid f^*(j) = g^*(j)\},$$

and let $p \colon \prod_{j \in J} h_j \to \prod_{j \in J'} h_j$ be the projection. We claim that p is the coequaliser of f and g in $\mathsf{Prod}(\mathsf{Field})$. First note that $pf = pg$, because $f^*p^* = g^*p^*$ by definition of J', and for $j \in J'$ we have $f_j = g_j$, because they both have the same inverse, i.e. $s_{f^*(j)} = s_{g^*(j)}$. Now let M be a set, l_m be a field for each $m \in M$, and $d \colon \prod_{j \in J} h_j \to \prod_{m \in M} l_m$ be a morphism in $\mathsf{Prod}(\mathsf{Field})$ such that $df = dg$. For each $m \in M$, we must have $f^*d^*(m) = g^*d^*(m)$, so $d^*(m) \in J'$. This shows that d factors uniquely through p.

Lastly, let us consider the coequaliser of f and g in Set. This is the quotient of $\prod_{j \in J} h_j$ by the equivalence relation generated by setting $f(x) \sim g(x)$ for all $x \in \prod_{i \in I} k_i$. It suffices to show that, for $y, y' \in \prod_{j \in J} h_j$,

$$y \sim y' \iff \forall j \in J',\ y_j = y'_j, \tag{6.8}$$

for then the quotient map is precisely p. First, note that for any $x \in \prod_{i \in I} k_i$ and $j \in J'$, we have

$$f(x)_j = f_j(x_{f^*(j)}) = g_j(x_{g^*(j)}) = g(x)_j.$$

Since the condition on the right-hand side of (6.8) is clearly an equivalence relation, we conclude that it contains $\sim$. Conversely, suppose $y_j = y'_j$ for all $j \in J'$. Let $x \in \prod_{i \in I} k_i$ be such that

$$x_i = \begin{cases} s(y)_i = s_i(y_{s^*(i)}) & \text{if } i \in \operatorname{im} f^*, \\ s(y')_i = s_i(y'_{s^*(i)}) & \text{if } i \in \operatorname{im} g^*, \\ 0 & \text{otherwise.} \end{cases}$$

This is well defined because if $i = f^*(j) = g^*(j')$ for some $j, j' \in J$, then applying s^* throughout gives $s^*(i) = j = j'$, and so $j \in J'$ and $y_j = y'_j$. For $j \in J$, we have

$$f(x)_j = f_j(x_{f^*(j)}) = f_j(s(y)_{f^*(j)}) = f(s(y))_j = y_j,$$

so $f(x) = y$. Similarly, $g(x) = y'$, and we conclude that $y \sim y'$ as needed. □

Theorem 6.3.6. *The category* $\mathsf{Prod}(\mathsf{Field})$ *is monadic over* Set*, and the corresponding monad is the codensity monad of the forgetful functor* $\mathsf{Field} \to \mathsf{Set}$.

Proof. The functor $U^{\mathsf{R}}U^{\mathsf{K}}$ is a right adjoint, and it reflects isomorphisms because both U^{R} and U^{K} do. Proposition 6.3.5 then completes the hypotheses of the Crude Monadicity Theorem. □

Corollary 6.3.7. *The category* $\mathsf{Prod}(\mathsf{Field})$ *is complete and cocomplete. Both* U^{K} *and* $U^{\mathsf{R}}U^{\mathsf{K}}$ *create all limits and reflexive coequalisers.*

Proof. Any category monadic over Set is complete and cocomplete by Theorem 2.4.5. The statement about limits follows from the fact that monadic functors create limits. They also create any colimits that the monad preserves. We have just shown that $U^{\mathsf{R}}U^{\mathsf{K}}$ preserves reflexive coequalisers, and U^{R} reflects them, so U^{K} also preserves them. □

This is quite remarkable: Field is a well-known example of a category that is not monadic over Set. This is clear because Field does not have products. Theorem 6.3.6 says that this is all it is missing. In the terminology of Section 1.2, we may summarise this with the phrase:

the smallest algebraic category containing the category
of fields is the category of products of fields.

Alternative perspectives on Theorem 6.3.6 are given by Kennison and Gildenhuys [37, pp. 341–342] and Diers [22, §3]. The former uses a topological approach: fields are given the discrete topology, and their products, the product topology. The morphisms in $\mathsf{Prod}(\mathsf{Field})$ are then precisely the continuous ring homomorphisms. Diers, on the other hand, shows that $\mathsf{Field} \to \mathsf{Set}$ is multimonadic and satisfies (D), and then applies Theorem 4.3.22.

We conclude this section by examining the operations in the theory of products of fields, and showing that the corresponding monad does not have rank. For ease of notation, we write $\mathsf{T} = U^{\mathsf{R}}\mathsf{K}F^{\mathsf{R}}$ for this monad. Let X be a set and

$$\theta \in \mathsf{T}X = \prod_{\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[X]} \operatorname{Frac}(\mathbb{Z}[X]/\mathfrak{p}).$$

Recall from Section 1.2 that we may think of θ as an X-ary operation in the theory of products of fields. Given a set I and fields k_i for each $i \in I$, θ gives an operation $\hat{\theta}\colon (\prod_{i\in I} k_i)^X \to \prod_{i\in I} k_i$ defined as follows:

> An element $a \in (\prod_{i\in I} k_i)^X$ corresponds to a unique ring homomorphism $f_a\colon \mathbb{Z}[X] \to \prod_{i\in I} k_i$. For each $i \in I$, $\ker \pi_i f_a$ is a prime ideal of $\mathbb{Z}[X]$. Then
>
> $$\hat{\theta}(a)_i = (\pi_i f_a)_{\ker \pi_i f_a}(\theta_{\ker \pi_i f_a}),$$
>
> where $(\pi_i f_a)_{\ker \pi_i f_a}\colon \operatorname{Frac}(\mathbb{Z}[X]/\ker \pi_i f_a) \to k_i$ is the map induced by $\pi_i f_a$.

This means that $\hat{\theta}$ will typically act differently on each component of the product. Moreover, for a fixed $a \in (\prod_{i\in I} k_i)^X$, only those components of θ indexed by $\ker \pi_i f_a$ for each $i \in I$ matter, in the sense that if $\zeta \in \mathsf{T}X$ is such that $\theta_{\ker \pi_i f_a} = \zeta_{\ker \pi_i f_a}$ for each i, then $\hat{\theta}(a) = \hat{\zeta}(a)$.

Examples 6.3.8.

(i) The constants of the theory of $\mathsf{Prod}(\mathsf{Field})$ are the elements of $\mathbb{Q} \times \prod_p \mathbb{Z}/p$, where the product is over all prime numbers p. When acting on a field k, only the component corresponding to $\operatorname{char} k$ matters.

(ii) The set of unary operations is $\prod_{\mathfrak{p}\in\operatorname{Spec}\mathbb{Z}[x]} \operatorname{Frac}(\mathbb{Z}[x]/\mathfrak{p})$. For a field k and $a \in k$, only the component corresponding to the kernel of the map $\mathbb{Z}[x] \to k$

sending x to a matters. For instance, for $\sqrt{2} \in \mathbb{R}$, the kernel is the principal ideal $(x^2 - 2)$, and then the corresponding component of a unary operation is an element of $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{R}$.

(iii) For $n \in \mathbb{N}$, consider the operation $\theta \in \mathsf{T}\{x_1, \dots, x_n\}$ that is 1 in the components corresponding to the prime ideals $(0), (2), (3), (5), \dots$ of $\mathbb{Z}[x_1, \dots, x_n]$, and zero elsewhere. For a field k and $a_1, \dots, a_n \in k$, we see that $\hat{\theta}(a_1, \dots, a_n) = 1$ iff the $a_1, \dots, a_n$ are algebraically independent over the prime subfield of k.

Proposition 6.3.9. *The codensity monad of the forgetful functor* $\mathsf{Field} \to \mathsf{Set}$ *has no rank.*

Proof. Let κ be a regular cardinal, equipped with its least well-order. The assignment $i \mapsto \mathrm{seg}(i)$ gives a functor $D\colon \kappa \to \mathsf{Set}$, whose colimit is κ itself. The fact that κ is regular implies that this is a κ-filtered colimit.

For $i \in \kappa$ and $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[\kappa]$, the $\mathfrak{p}$-component of the comparison map $\mathsf{T}Di \to \mathsf{T}\kappa$ is the bottom arrow in the square

$$\begin{tikzcd}
\mathbb{Z}[Di] \arrow[r, hook] \arrow[d, "\mathrm{can}"'] & \mathbb{Z}[\kappa] \arrow[d, "\mathrm{can}"] \\
\mathrm{Frac}(\mathbb{Z}[Di]/(\mathfrak{p} \cap \mathbb{Z}[Di])) \arrow[r, hook] & \mathrm{Frac}(\mathbb{Z}[\kappa]/\mathfrak{p})
\end{tikzcd}$$

Seeing Di as a subset of κ, let $\mathfrak{p}_i$ be the ideal generated by $\eta^{\mathsf{R}}_{\kappa} Di \subseteq \mathbb{Z}[\kappa]$. This is prime, since $\mathbb{Z}[\kappa]/\mathfrak{p}_i$ is the integral domain $\mathbb{Z}[\uparrow i]$, where we write $\uparrow i$ for the set of elements strictly above i in κ. In the $\mathfrak{p}_i$-component, the bottom map in the square becomes $\mathbb{Q} \to \mathbb{Q}(\uparrow i)$.

Using (6.6), let $y \in \mathsf{K}\mathbb{Z}[\kappa]$ be such that

$$y_{\mathfrak{p}} = \begin{cases} i \in \mathbb{Q}(\uparrow i) & \text{if } \mathfrak{p} = \mathfrak{p}_i \text{ for some } i \in \kappa, \\ 0 \in \mathrm{Frac}(\mathbb{Z}[\kappa]/\mathfrak{p}) & \text{otherwise,} \end{cases}$$

for each $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}[\kappa]$. An element of $\mathrm{colim}_{\kappa} \mathsf{T}D$ is represented by some $x \in \mathsf{K}(\mathbb{Z}[Dj])$ for some $j \in \kappa$. The image of such an x cannot be y, because if $i > j$, its $\mathfrak{p}_i$-component is a rational constant, and not i. Thus, the comparison map $\mathrm{colim}_{\kappa} \mathsf{T}D \to \mathsf{T}(\mathrm{colim}_{\kappa} D)$ is not surjective, and T does not preserve κ-filtered colimits. □

We end by remarking that a similar story unfolds if one takes the inclusion of the category Field_p of fields of characteristic p into CRing, where p is zero or a prime number. In this case, the categories of algebras for the codensity monads of the forgetful functors into both CRing and Set are the free product completion $\mathsf{Prod}(\mathsf{Field}_p)$.

6.4 Pushing forward along a monad

In this last section we consider the situation of pushing a monad forward along another monad. As the first nontrivial example of this, we compute the codensity monad of the exception monads E_E on Set for a not-necessarily-finite set E (Example 2.4.12).

Let t and s be monads on a 0-cell x of a 2-category. At first glance, one might not expect anything special to happen when taking $s_\sharp t$, since s here is only acting as a 1-cell. However, when s has a monad structure, there is some interaction between this monad and the pushforward:

Proposition 6.4.1. *With t and s as above, there is a canonical monad map $s \to s_\sharp t$ whenever the latter exists.*

Proof. Recall from Example 2.3.4 that

$$\begin{tikzcd}(x,t) \arrow[r, "(1_x, \eta^t)"] & (x, 1_x) \arrow[r, "(s, \mu^s)"] & (x, s)\end{tikzcd}$$

are lax monad functors. Their composite is hence a lax monad functor whose underlying 1-cell is s. By Theorem 4.1.5, this corresponds to a monad map $s \to s_\sharp t$. □

Note that $s_\sharp t$ does not depend on the monad structure on s, so any such monad structure will give a map of monads $s \to s_\sharp t$. For instance, let M be a set, and M_M be the functor $M \times -\colon \mathsf{Set} \to \mathsf{Set}$. Every monoid structure on M can be used to give M_M two monad structures whose algebras are, respectively, the corresponding left and right M-sets. An $(\mathsf{M}_M)_\sharp 1$-algebra will simultaneously have an underlying left and right M-set structure for every monoid structure on M.

The difficulty of computing pushforwards along a functor greatly increases with the complexity of the functor. Because of this, we will only consider the most basic examples.

Example 6.4.2. Let T and T_0 be the terminal and subterminal monads on Set of Examples 2.4.9 and 2.4.10. From (4.6), it is straightforward to see that

$$\mathsf{T}_\sharp 1 \cong \mathsf{T}, \quad \mathsf{T}_\sharp \mathsf{T}_0 \cong \mathsf{T}, \quad \mathsf{T}_\sharp \mathsf{T} \cong \mathsf{T}, \quad \mathsf{T}_{0\sharp} 1 \cong \mathsf{T}_0, \quad \mathsf{T}_{0\sharp} \mathsf{T}_0 \cong \mathsf{T}_0, \quad \text{and} \quad \mathsf{T}_{0\sharp} \mathsf{T} \cong \mathsf{T}.$$

The rest of this section is devoted to describing $(\mathsf{E}_E)_\sharp 1$ and its category of algebras for a nonempty set E. In fact, Diers's theory of multimonadic functors already answered this question whenever $|E| \geq 2$ (Example 4.3.23(ii)). We record this as a theorem here.

Theorem 6.4.3. *Let E be a set with at least two elements. The functor* $\mathsf{Prod}(\mathsf{Set}) \to \mathsf{Set}$ *given by*

$$(X_i)_{i \in I} \longmapsto \prod_{i \in I} (X_i + E)$$

is monadic, and the corresponding monad is the codensity monad of E_E.

It remains to describe $(\mathsf{E}_1)_\sharp 1$ and its category of algebras. For notational convenience, we will write E instead of E_1. Since the functor E does not satisfy (D), we will need to take a more careful approach. Coincidentally, the monad E is quite special: it is one of only six families of finitary monads on Set all of whose algebras are free. Four of these families are given in Part II of a theorem of Givant [29, p. 2]; the other two are the inconsistent monads T and T_0.

Let X be a set, and $A \subseteq X$. Throughout the remainder of this section we write $[A]\colon X \to \mathsf{E}A$ for the function that is the identity on A and sends everything else to the basepoint $* \in 1$. This is the initial object of the connected component of $X{\downarrow}\mathsf{E}$ corresponding to the map $\mathsf{E}! \cdot [A]\colon X \to \mathsf{E}1$, as described in Example 4.3.4(v). Using the limit formula (4.6), we get

$$\mathsf{E}_\sharp 1(X) = \prod_{A \subseteq X} \mathsf{E}A.$$

For $A \subseteq X$, we write $\langle A\rangle\colon \mathsf{E}_\sharp 1(X) \to \mathsf{E}A$ for the corresponding product projection. Given a function $f\colon X \to Y$ and $B \subseteq Y$ we have

$$\langle B\rangle \cdot \mathsf{E}_\sharp 1(f) = \mathsf{E}(f|_{f^{-1}B}) \cdot \langle f^{-1}B\rangle. \tag{6.9}$$

The counit $\epsilon_X\colon \mathsf{E}_\sharp 1(\mathsf{E}X) \to \mathsf{E}X$ is simply $\langle X\rangle$, the projection onto the component corresponding to $X \subseteq \mathsf{E}X$. For each $A \subseteq X$, the unit η of $\mathsf{E}_\sharp 1$ satisfies

$$\langle A\rangle \cdot \eta_X = [A], \tag{6.10}$$

while the multiplication μ is determined by

$$\langle A\rangle \cdot \mu_X = \mathsf{E}\left(\langle A\rangle|_{\langle A\rangle^{-1}A}\right) \cdot \langle\langle A\rangle^{-1}A\rangle. \tag{6.11}$$

We have dropped the name of the monad $\mathsf{E}_\sharp 1$ from the notation of the unit and multiplication for convenience. No confusion should arise since this is the only monad we consider for the rest of the section.

As with the multiplication of K in Section 6.3, note that μ_X only depends on the components indexed by $\langle A\rangle^{-1}A$ for $A \subseteq X$. This will allow us to use a similar strategy as in Proposition 6.3.2 to show that the $\mathsf{E}_\sharp 1$-algebras are products of pointed sets. First, we show how to endow a product of pointed sets with an $\mathsf{E}_\sharp 1$-algebra structure.

Proposition 6.4.4. *Let I be a set, and X_i be nonempty sets for each $i \in I$. Let $X = \prod_{i\in I} \mathsf{E}X_i$, and $Y_i = \pi_i^{-1}X_i \subseteq X$ for each $i \in I$. Then the unique function $\zeta\colon \mathsf{E}_\sharp 1(X) \to X$ such that*

$$\pi_i\zeta = \mathsf{E}(\pi_i|_{Y_i}) \cdot \langle Y_i\rangle \tag{6.12}$$

for each $i \in I$ is an $\mathsf{E}_\sharp 1(X)$-algebra structure map.

Proof. First consider the case where $I = \{1\}$. The comparison functor $\mathsf{Set} \to \mathsf{Set}^{\mathsf{E}_\sharp 1}$ over E sends a set X_1 to the $\mathsf{E}_\sharp 1$-algebra $\epsilon_{X_1}\colon \mathsf{E}_\sharp 1(\mathsf{E}X_1) \to \mathsf{E}X_1$. This agrees with the one in the statement because ϵ_{X_1} is just $\langle X_1\rangle$. For a general I, taking the product in $\mathsf{Set}^{\mathsf{E}_\sharp 1}$ of each of these $\mathsf{E}_\sharp 1$-algebra structures on the $\mathsf{E}X_i$ gives the desired $\mathsf{E}_\sharp 1$-algebra structure map ζ on X. □

The result of the previous proposition remains true even if some of the X_i are empty. Asking that they be nonempty is a normalisation condition, since if $X_i = \varnothing$ then removing it does not change the set X or its $\mathsf{E}_\sharp 1$-algebra structure up to isomorphism. Note that we can still obtain a singleton algebra from this proposition by taking $I = \varnothing$.

Our plan is to show that every $\mathsf{E}_\sharp 1$-algebra is isomorphic to one given by Proposition 6.4.4. In order to do this, we must identify the subsets that will play the role of the Y_i there.

Definition 6.4.5. Let (X, ξ) be an $\mathsf{E}_\sharp 1$-algebra. A subset $A \subseteq X$ is **relevant** if it is nonempty and $\xi^{-1}A = \langle A\rangle^{-1}A$ as subsets of $\mathsf{E}_\sharp 1(X)$. Let $R(\xi)$ be the set of all relevant subsets.

Lemma 6.4.6. *Let (X, ξ) be an $\mathsf{E}_\sharp 1$-algebra, and $A, B \subseteq X$. If $\xi^{-1}B = \langle A\rangle^{-1}A$, then $A = B$.*

Proof. For $x \in X$, we have

$$x \in A \overset{(6.10)}{\iff} \eta_X(x) \in \langle A\rangle^{-1}A = \xi^{-1}B \iff \xi\eta_X(x) = x \in B. \qquad \square$$

The name 'relevant' is motivated by the next lemma.

Lemma 6.4.7. *Let (X, ξ) be an $\mathsf{E}_\sharp 1$-algebra. Then ξ factors through the projection*

$$\tau \colon \mathsf{E}_\sharp 1(X) \longrightarrow \prod_{A \in R(\xi)} \mathsf{E}A.$$

Proof. Let $y, y' \in \mathsf{E}_\sharp 1(X)$. It suffices to show that $\xi(y) = \xi(y')$ whenever $\langle A\rangle(y) = \langle A\rangle(y')$ for all $A \in R(\xi)$. We will do this by constructing $z, z' \in (\mathsf{E}_\sharp 1)^2(X)$ such that $\mu_X(z) = y$ and $\mu_X(z') = y'$, and yet $\mathsf{E}_\sharp 1(\xi)(z) = \mathsf{E}_\sharp 1(\xi)(z')$. The multiplication axiom for ξ then implies

$$\xi(y) = \xi\mu_X(z) = \xi \cdot \mathsf{E}_\sharp 1(\xi)(z) = \xi \cdot \mathsf{E}_\sharp 1(\xi)(z') = \xi\mu_X(z') = \xi(y').$$

We set

$$\langle\langle A\rangle^{-1}A\rangle(z) = \langle A\rangle(y) \qquad \text{and} \qquad \langle\langle A\rangle^{-1}A\rangle(z') = \langle A\rangle(y'),$$

for $A \subseteq X$, and $\langle B\rangle(z) = \langle B\rangle(z') = *$ for any other $B \subseteq \mathsf{E}_\sharp 1(X)$. That $\mu_X(z) = y$ and $\mu_X(z') = y'$ is immediate from (6.11). If $A \subseteq X$ is relevant, then

$$\langle\xi^{-1}A\rangle(z) = \langle\langle A\rangle^{-1}A\rangle(z) = \langle A\rangle(y) = \langle A\rangle(y') = \langle\langle A\rangle^{-1}A\rangle(z') = \langle\xi^{-1}A\rangle(z'),$$

and otherwise $\langle\xi^{-1}A\rangle(z) = * = \langle\xi^{-1}A\rangle(z')$ by Lemma 6.4.6. It follows from (6.9) that $\mathsf{E}_\sharp 1(\xi)(z) = \mathsf{E}_\sharp 1(\xi)(z')$. $\square$

The relevant subsets of the $\mathsf{E}_\sharp 1$-algebra (X, ζ) induced by a family of nonempty sets $(X_i)_{i\in I}$ by Proposition 6.4.4 are exactly the Y_i defined there for $i \in I$. Indeed, given $i \in I$, Y_i is nonempty because X_i is, and for $y \in \mathsf{E}_\sharp 1(X)$ we have

$$\zeta(y) \in Y_i \iff \pi_i\zeta(y) = \mathsf{E}(\pi_i|_{Y_i}) \cdot \langle Y_i\rangle(y) \in X_i \iff \langle Y_i\rangle(y) \in Y_i,$$

so that $\zeta^{-1}Y_i = \langle Y_i\rangle^{-1}Y_i$. Conversely, if $A \subseteq X$ is nonempty and not equal to one of the Y_i, then the value of $\zeta(y)$ does not depend on $\langle A\rangle(y)$. For $x \in A$, let $y \in \mathsf{E}_\sharp 1(X)$ be equal to $\eta_X(x)$ in all components but the one corresponding to A, where instead $\langle A\rangle(y) = *$. Then $\zeta(y) = x \in A$ but $\langle A\rangle(y) \notin A$, so that A is not relevant.

We will need some notation for the proof of the next proposition. Let (X,ξ) be an $\mathsf{E}_\sharp 1$-algebra. Let $\widetilde{\xi}$ be the factorisation of ξ through τ afforded by Lemma 6.4.7. For $A \subseteq X$ and $x \in X$, let δ^x_A be the unique element of $\mathsf{E}_\sharp 1(X)$ such that

$$\langle A\rangle(\delta^x_A) = [A](x) \qquad \text{and} \qquad \langle B\rangle(\delta^x_A) = *$$

for all other $B \subseteq X$. In particular, if $x \notin A$, all components of δ^x_A are $*$.

Proposition 6.4.8. *Every $\mathsf{E}_\sharp 1$-algebra is isomorphic to one described by Proposition 6.4.4.*

Proof. Let (X,ξ) be an $\mathsf{E}_\sharp 1$-algebra and, for $A \in R(\xi)$, let

$$X_A = A \setminus \left(\bigcup_{B \in R(\xi)\setminus\{A\}} B \right) \subseteq X.$$

Each X_A is nonempty. Indeed, if A were a subset of a union $\bigcup_{i\in I} B_i$ of relevant subsets B_i, then

$$\langle A\rangle^{-1}A = \xi^{-1}A \subseteq \bigcup_{i\in I} \xi^{-1}B_i = \bigcup_{i\in I} \langle B_i\rangle^{-1}B_i.$$

This is clearly impossible, because the values of $\langle A\rangle(y)$ and $\langle B_i\rangle(y)$ for $y \in \mathsf{E}_\sharp 1(X)$ are independent.

Let $Y = \prod_{A\in R(\xi)} \mathsf{E}X_A$ with its canonical $\mathsf{E}_\sharp 1$-algebra structure map ζ given by Proposition 6.4.4. We show that (X,ξ) and (Y,ζ) are isomorphic. The inclusions $X_A \subseteq A$ for each A, give an inclusion $Y \subseteq \prod_{A\in R(\xi)} \mathsf{E}A$. Our isomorphism will be the composite

$$\beta\colon Y \hookrightarrow \prod_{A\in R(\xi)} \mathsf{E}A \xrightarrow{\widetilde{\xi}} X.$$

Its inverse is the unique map $\gamma\colon X \to Y$ such that

$$\pi_A\gamma(x) = \begin{cases} \xi(\delta^x_A) & \text{if } x \in A, \\ * & \text{otherwise,} \end{cases} \tag{6.13}$$

for all $x \in X$ and $A \in R(\xi)$. To see that $\xi(\delta^x_A) \in X_A$ when $x \in A$, note that for any $B \in R(\xi)$ we have

$$\xi(\delta^x_A) \in B \iff \delta^x_A \in \xi^{-1}B = \langle B\rangle^{-1}B \iff \langle B\rangle(\delta^x_A) \in B \iff A = B.$$

Given $x \in X$, we need $\beta\gamma(x) = x$. Let $z \in (\mathsf{E}_\sharp 1)^2(X)$ be such that $\langle \xi^{-1}A\rangle(z) = \delta^x_A$ for each $A \in R(\xi)$. From (6.9) and (6.11), we see that

$$\tau \cdot \mathsf{E}_\sharp 1(\xi)(z) = \gamma(x) \qquad \text{and} \qquad \tau\mu_X(z) = \tau\eta_X(x).$$

Then

$$\begin{aligned}\beta\gamma(x) &= \widetilde{\xi}\tau \cdot \mathsf{E}_\sharp 1(\xi)(z) \\ &= \xi \cdot \mathsf{E}_\sharp 1(\xi)(z) \\ &= \xi\mu_X(z) && \text{by the multiplication axiom} \\ &= \widetilde{\xi}\tau\mu_X(z) \\ &= \widehat{\xi}\tau\eta_X(x) \\ &= \xi\eta_X(x) \\ &= x && \text{by the unit axiom.}\end{aligned}$$

Given $y \in Y$, we also need $\gamma\beta(y) = y$. Let $z \in \mathsf{E}_\sharp 1(X)$ be such that $\langle A\rangle(z) = y_A$ for all $A \in R(\xi)$, e.g. by setting $\langle B\rangle(z) = *$ for any nonrelevant $B \subseteq X$. Then $\beta(y) = \xi(z)$ by definition of β. If y_A is in $X_A \subseteq A$ for some $A \in R(\xi)$, then $z \in \langle A\rangle^{-1}A = \xi^{-1}A$, so

$$\mathsf{E}_\sharp 1(\xi)(\delta^z_{\xi^{-1}A}) = \delta^{\xi(z)}_A \qquad \text{and} \qquad \mu_X(\delta^z_{\xi^{-1}A}) = \delta^{y_A}_A$$

by (6.9), (6.11) and Lemma 6.4.6. Moreover, $\tau\eta_X(y_A) = \tau(\delta^{y_A}_A)$ because $y_A \in X_A$ implies that $y_A \notin B$ for any other $B \in R(\xi)$. Then

$$\begin{aligned}\pi_A\gamma\beta(y) &= \pi_A\gamma\xi(z) \\ &= \xi(\delta^{\xi(z)}_A) && \text{by (6.13) since } z \in \xi^{-1}A \\ &= \xi \cdot \mathsf{E}_\sharp 1(\xi)(\delta^z_{\xi^{-1}A}) \\ &= \xi\mu_X(\delta^z_{\xi^{-1}A}) && \text{by the multiplication axiom} \\ &= \xi(\delta^{y_A}_A) \\ &= \widetilde{\xi}\tau(\delta^{y_A}_A) \\ &= \widetilde{\xi}\tau\eta_X(y_A) \\ &= \xi\eta_X(y_A) \\ &= y_A && \text{by the unit axiom.}\end{aligned}$$

If on the other hand $y_A = *$, then $z \notin \langle A\rangle^{-1}A = \xi^{-1}A$, so

$$\pi_A\gamma\beta(y) = \pi_A\gamma\xi(z) = *$$

by (6.13).

Lastly, we show that β is an $\mathsf{E}_\sharp 1$-algebra map. For $A \in R(\xi)$, let $Y_A = \pi_A^{-1}X_A \subseteq Y$ as in Proposition 6.4.4, and note that $Y_A = \widetilde{\xi}^{-1}A \cap Y = \beta^{-1}A$. Let $z \in \mathsf{E}_\sharp 1(Y)$. By (6.12) and (6.9),

$$\pi_A\zeta(z) = \mathsf{E}(\pi_A|_{Y_A}) \cdot \langle Y_A\rangle(z) \qquad \text{and} \qquad \langle A\rangle \cdot \mathsf{E}_\sharp 1(\beta)(z) = \mathsf{E}(\beta|_{Y_A}) \cdot \langle Y_A\rangle(z)$$

for all $A \in R(\xi)$. Let $w \in (\mathsf{E}_\sharp 1)^2(X)$ be such that

$$\langle A\rangle \cdot \langle \xi^{-1}A\rangle(w) = \delta^{\zeta(x)_A}_A$$

for all $A \in R(\xi)$, where $\pi_A\zeta(x) \in \mathsf{E}X_A \subseteq \mathsf{E}A$. Then from (6.12), (6.11) and (6.9), we have

$$\langle A \rangle \cdot \mu_X(w) = \pi_A\zeta(z) \qquad \text{and} \qquad \tau \cdot \mathsf{E}_\sharp 1(\xi)(w) = \tau \cdot \mathsf{E}_\sharp 1(\beta)(z).$$

Altogether,

$$\begin{aligned}
\beta\zeta(z) &= \xi\mu_X(w) && \text{by definition of } \beta \\
&= \xi \cdot \mathsf{E}_\sharp 1(\xi)(w) && \text{by the multiplication axiom} \\
&= \tilde{\xi}\tau \cdot \mathsf{E}_\sharp 1(\xi)(w) \\
&= \tilde{\xi}\tau \cdot \mathsf{E}_\sharp 1(\beta)(z) \\
&= \xi \cdot \mathsf{E}_\sharp 1(\beta)(z).
\end{aligned}$$

□

This characterises the $\mathsf{E}_\sharp 1$-algebras, but we still need to study the maps between them. That is the content of the next proposition. Given a family of nonempty sets $(X_i)_{i \in I}$ and $x \in X_i$ for some $i \in I$, we now write δ_i^x for the unique element of $\prod_{i \in I} \mathsf{E}X_i$ with x in its i-th component and $*$ elsewhere.

Proposition 6.4.9. *Let I and J be sets, X_i and Y_j be nonempty sets for each $i \in I$ and $j \in J$, and $X = \prod_{i \in I} \mathsf{E}X_i$ and $Y = \prod_{j \in J} \mathsf{E}Y_j$. Let (X, ξ) and (Y, ζ) be the $\mathsf{E}_\sharp 1$-algebras given by Proposition 6.4.4. A function $f \colon X \to Y$ is an $\mathsf{E}_\sharp 1$-algebra map $(X, \xi) \to (Y, \zeta)$ iff there exist*

- *a function $f^* \colon J \to \mathsf{E}I$; and*
- *for each $j \in J$ such that $f^*(j) \in I$, a function $f_j \colon X_{f^*(j)} \to Y_j$,*

such that

$$\pi_j f = \begin{cases} \mathsf{E}f_j \cdot \pi_{f^*(j)} & \textit{whenever } f^*(j) \in I, \\ * & \textit{otherwise.} \end{cases}$$

Moreover, the data of f^* and the f_j whenever $f^*(j) \in I$ uniquely determines f.

Proof. Let f be an $\mathsf{E}_\sharp 1$-algebra map. For $j \in J$, define $f^*(j)$ to be the (unique) i such that $(\pi_j f)^{-1} Y_j = \pi_i^{-1} X_i$, or $*$ if no such i exists. Since f is an $\mathsf{E}_\sharp 1$-algebra map,

$$\begin{aligned}
\pi_j f \xi &= \pi_j \zeta \cdot \mathsf{E}_\sharp 1(f) \\
&= \mathsf{E}(\pi_j|_{\pi_j^{-1} Y_j}) \cdot \langle \pi_j^{-1} Y_j \rangle \cdot \mathsf{E}_\sharp 1(f) && \text{by (6.12)} \\
&= \mathsf{E}(\pi_j f|_{(\pi_j f)^{-1} Y_j}) \cdot \langle (\pi_j f)^{-1} Y_j \rangle && \text{by (6.9).}
\end{aligned} \tag{6.13}$$

Given $x \in X$, define $z \in \mathsf{E}_\sharp 1(X)$ by putting $\langle \pi_i^{-1} X_i \rangle (z) = \delta_i^{x_i}$ for each $i \in I$, and $\langle A \rangle (z) = *$ for any other $A \subseteq X$, so that $\xi(z) = x$ by (6.12). If $f^*(j) = i \in I$, (6.13) gives

$$\pi_j f(x) = \mathsf{E}(\pi_j f|_{\pi_i^{-1} X_i})(\delta_i^{x_i}).$$

Since the right-hand side only depends on x_i, it is clear that $\pi_j f$ factors through π_i. This factorisation is unique because π_i is epic. Moreover,

$$\pi_j f(x) = * \iff \delta_i^{x_i} \notin \pi_i^{-1} X_i \iff x_i \notin X_i,$$

so the factorisation is of the form $\mathsf{E}f_j$ for a unique $f_j\colon X_i \to Y_j$. If on the other hand $f^*(j) = *$, then (6.13) gives $\pi_j f(x) = *$, as needed.

Now suppose f satisfies the conditions in the statement. We need to show both sides of (6.13) are equal for all $j \in J$ and $z \in \mathsf{E}_\sharp 1(X)$. If $f^*(j) = i \in I$, then

$$(\pi_j f)^{-1} Y_j = (\mathsf{E}f_j \cdot \pi_i)^{-1} Y_j = \pi_i^{-1} X_i,$$

and the left-hand side becomes

$$\begin{aligned} \mathsf{E}f_j \cdot \pi_i \xi(z) &= \mathsf{E}f_j \cdot \mathsf{E}(\pi_i|_{\pi_i^{-1}X_i}) \cdot \langle \pi_i^{-1} X_i \rangle (z) && \text{by (6.12)} \\ &= \mathsf{E}(f_j \pi_i|_{\pi_i^{-1}X_i}) \cdot \langle \pi_i^{-1} X_i \rangle (z), \end{aligned}$$

which equals the right-hand side. If $f^*(j) = *$, then $(\pi_j f)^{-1} Y_j = \varnothing$, and so the only thing $\langle (\pi_j f)^{-1} Y_j \rangle (z)$ can be is $*$. □

Altogether, our results give:

Theorem 6.4.10. *The category* $\mathsf{Set}^{\mathsf{E}_\sharp 1}$ *is equivalent to the category whose objects are products of nonempty sets with a basepoint adjoined and whose morphisms are those described in Proposition 6.4.9.*

Note how this is similar to the situation with K in Section 6.3. Crucially, however, $\mathsf{Set}^{\mathsf{E}_\sharp 1}$ is not a product completion of $\mathsf{Set}^{\mathsf{E}}$. Not only this, but there is no functor $\mathsf{Set}^{\mathsf{E}} \to \mathsf{Set}^{\mathsf{E}_\sharp 1}$ over Set. Indeed, suppose G were such a functor. Let $X = \{x, y, z\}$ be an E-algebra with basepoint z, and $f\colon X \to X$ be the unique E-algebra map sending x to y and y to z. Since $|X|$ is prime, the set X has a unique-up-to-isomorphism $\mathsf{E}_\sharp 1$-algebra structure, isomorphic to the $\mathsf{E}_\sharp 1$-algebra $\mathsf{E}2$ of Proposition 6.4.4. For $f\colon GX \to GX$ to be an $\mathsf{E}_\sharp 1$-algebra map, it must fix the new basepoint, which must then be z. But f is not constant at z, nor of the form $\mathsf{E}f'$ for some function $f'\colon \{x, y\} \to \{x, y\}$, so it cannot satisfy the conditions of Proposition 6.4.9.

Index of diagram labels

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